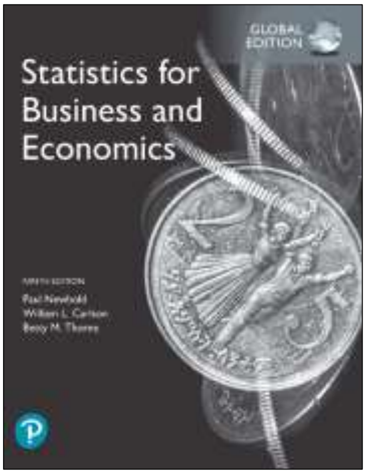
Statistics for Business and Economics

Ninth Edition, Global Edition



Chapter 7
Estimation: Single Population

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Chapter Goals

After completing this chapter, you should be able to:

- Distinguish between a point estimate and a confidence interval estimate
- Construct and interpret a confidence interval estimate for a single population mean using both the *Z* and *t* distributions
- Form and interpret a confidence interval estimate for a single population proportion
- Create confidence interval estimates for the variance of a normal population
- Determine the required sample size to estimate a mean or proportion within a specified margin of error

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Confidence Intervals (1 of 2)

Contents of this chapter:

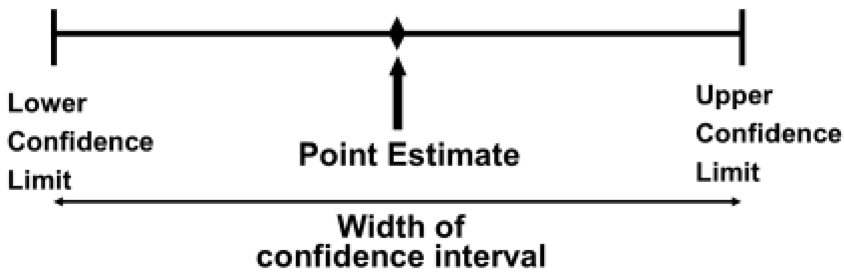
- Confidence Intervals for the Population Mean, μ
  - when Population Variance σ^2 is Known
  - when Population Variance σ^2 is Unknown
- Confidence Intervals for the Population Proportion, P (large samples)
- Confidence interval estimates for the variance of a normal population
- Finite population corrections
- Sample-size determination

Section 7.1 Properties of Point Estimators

- An estimator of a population parameter is
  - a random variable that depends on sample information . . .
  - whose value provides an approximation to this unknown parameter
- A specific value of that random variable is called an estimate

Point and Interval Estimates

- A point estimate is a single number,
- a confidence interval provides additional information about variability



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Point Estimates

We can estimate a Population Parameter …		with a Sample Statistic (a Point Estimate)
Mean	μ	$\bar{x}$
Proportion	P	$\hat{p}$

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Unbiasedness (1 of 2)

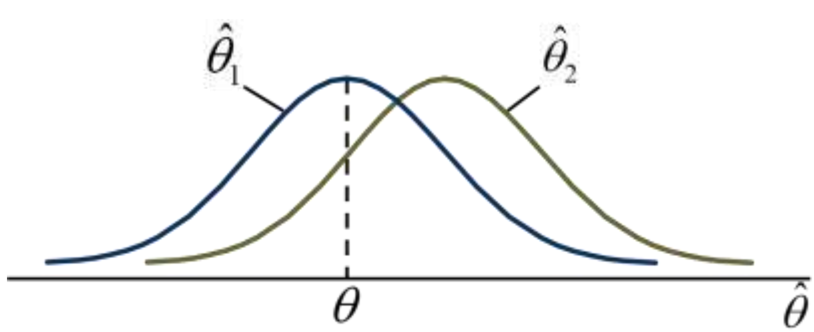
- A point estimator $\hat{\theta}$ is said to be an unbiased estimator of the parameter $\boldsymbol{\theta}$ if its expected value is equal to that parameter:

$$E(\hat{\theta}) = \theta$$

- Examples:
  - The sample mean $\bar{x}$ is an unbiased estimator of μ
  - The sample variance s^2 is an unbiased estimator σ^2
  - The sample proportion $\hat{p}$ is an unbiased estimator of *P*

Unbiasedness (2 of 2)

- $\hat{\theta}_1$ is an unbiased estimator, $\hat{\theta}_2$ is biased:



Bias

- Let $\hat{\theta}$ be an estimator of $\boldsymbol{\theta}$
- The bias in $\hat{\theta}$ is defined as the difference between its mean and $\boldsymbol{\theta}$

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

- The bias of an unbiased estimator is 0

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Most Efficient Estimator

- Suppose there are several unbiased estimators of $\boldsymbol{\theta}$
- The most efficient estimator or the minimum variance unbiased estimator of $\boldsymbol{\theta}$ is the unbiased estimator with the smallest variance
- Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two unbiased estimators $\boldsymbol{\theta}$, based on the same number of sample observations. Then,
  - $\hat{\theta}_1$ is said to be more efficient than $\hat{\theta}_2$ if $Var(\hat{\theta}_1) < Var(\hat{\theta}_2)$
  - The relative efficiency of $\hat{\theta}_1$ with respect to $\hat{\theta}_2$ is the ratio of their variances:

$$\text{Relative Efficiency} = \frac{Var(\hat{\theta}_2)}{Var(\hat{\theta}_1)}$$

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Confidence Interval Estimation

- How much uncertainty is associated with a point estimate of a population parameter?
- An interval estimate provides more information about a population characteristic than does a point estimate
- Such interval estimates are called confidence interval estimates

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Confidence Interval Estimate

- An interval gives a range of values:
  - Takes into consideration variation in sample statistics from sample to sample
  - Based on observation from 1 sample
  - Gives information about closeness to unknown population parameters
  - Stated in terms of level of confidence
    - Can never be 100% confident

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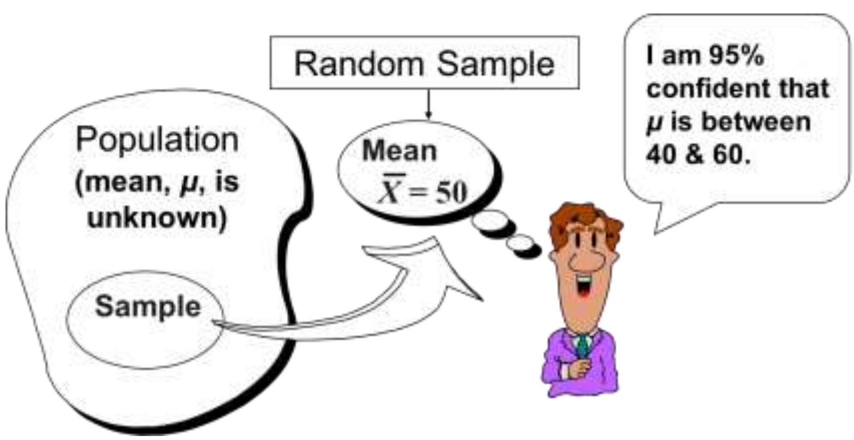
Confidence Interval and Confidence Level

- If $P(a < \theta < b) = 1 - \alpha$ then the interval from *a* to *b* is called a $100(1-\alpha)\%$ confidence interval of θ.
- The quantity $100(1-\alpha)\%$ is called the confidence level of the interval
  - α is between 0 and 1
  - In repeated samples of the population, the true value of the parameter θ would be contained in $100(1-\alpha)\%$ of intervals calculated this way.
  - The confidence interval calculated in this manner is written as $a < \theta < b$ with $100(1-\alpha)\%$ confidence

Estimation Process



Confidence Level, Left Parenthesis 1 Minus Alpha Right Parenthesis

- Suppose confidence level = 95%
- Also written $(1-\alpha) = 0.95$
- A relative frequency interpretation:
  - From repeated samples, 95% of all the confidence intervals that can be constructed of size *n* will contain the unknown true parameter
- A specific interval either will contain or will not contain the true parameter
  - No probability involved in a specific interval

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General Formula

- The general form for all confidence intervals is:

$$\hat{\theta} \pm ME$$

$$\textbf{Point Estimate} \pm \textbf{Margin of Error}$$

- The value of the margin of error depends on the desired level of confidence

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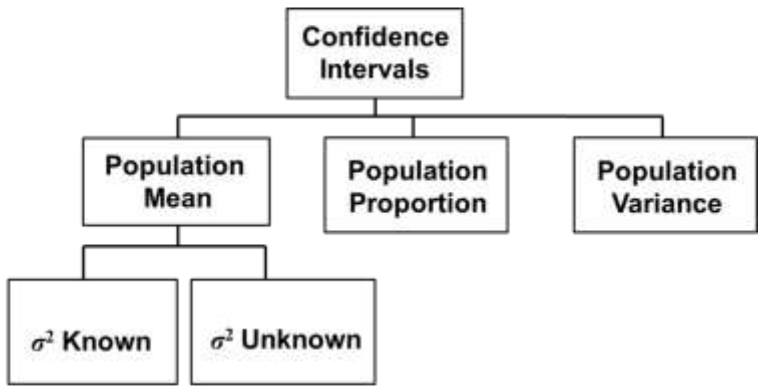
Confidence Intervals (2 of 2)



(From normally distributed populations)

Section 7.2 Confidence Interval Estimation for the Mean (Sigma Squared Known)

- Assumptions
  - Population variance σ^2 is known
  - Population is normally distributed
  - If population is not normal, use large sample
- Confidence interval estimate:

$$\bar{x} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

(where $z_{\frac{\alpha}{2}}$ is the normal distribution value for a probability of $\frac{\alpha}{2}$ in each tail)

Confidence Limits

- The confidence interval is

$$\bar{x} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

- The endpoints of the interval are

$$\text{UCL} = \bar{x} + z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$ Upper confidence limit

$$\text{LCL} = \bar{x} - z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$ Lower confidence limit

Margin of Error (1 of 2)

- The confidence interval,

$$\bar{x} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

- Can also be written as $\bar{x} \pm ME$
  where *ME* is called the margin of error

$$ME = z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

- The interval width, *w*, is equal to twice the margin of error

Reducing the Margin of Error

$$ME = z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

The margin of error can be reduced if

- the population standard deviation can be reduced $(\sigma \downarrow)$
- The sample size is increased $(n \uparrow)$
- The confidence level is decreased, $(1-\alpha) \downarrow$

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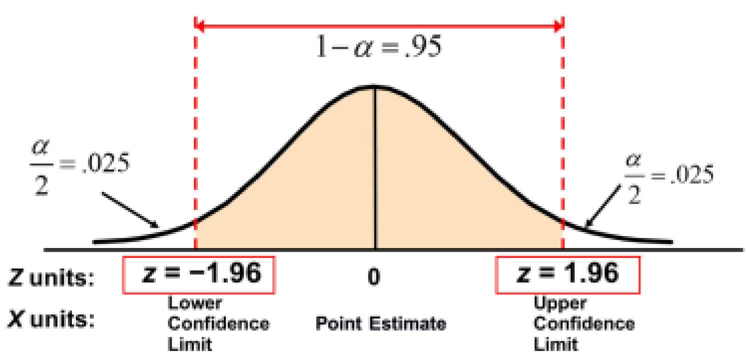
Finding Z of Start Expression Start Fraction Alpha over 2 End Fraction End Expression

- Consider a 95% confidence interval:



- Find $z_{.025} = \pm 1.96$ from the standard normal distribution table

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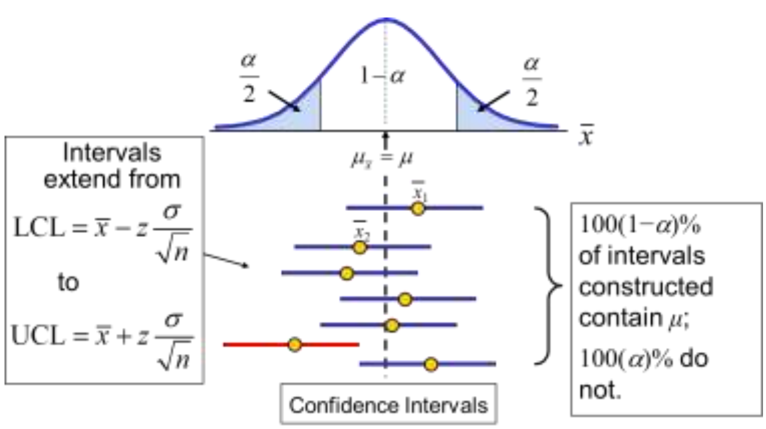
Common Levels of Confidence

- Commonly used confidence levels are 90%, 95%, 98%, and 99%

Confidence Level	Confidence Coefficient, $1-\alpha$	$Z_{\frac{\alpha}{2}}$ value
80%	.80	1.28
90%	.90	1.645
95%	.95	1.96
98%	.98	2.33
99%	.99	2.58
99.8%	.998	3.08
99.9%	.999	3.27

Intervals and Level of Confidence

Sampling Distribution of the Mean



Example 1 (1 of 2)

- A sample of 11 circuits from a large normal population has a mean resistance of 2.20 ohms. We know from past testing that the population standard deviation is 0.35 ohms.
- Determine a 95% confidence interval for the true mean resistance of the population.

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Example 1 (2 of 2)

- A sample of 11 circuits from a large normal population has a mean resistance of 2.20 ohms. We know from past testing that the population standard deviation is .35 ohms.
- Solution:

$$\bar{x} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

$$= 2.20 \pm 1.96\left(\frac{.35}{\sqrt{11}}\right)$$

$$= 2.20 \pm .2068$$

$$1.9932 < \mu < 2.4068$$

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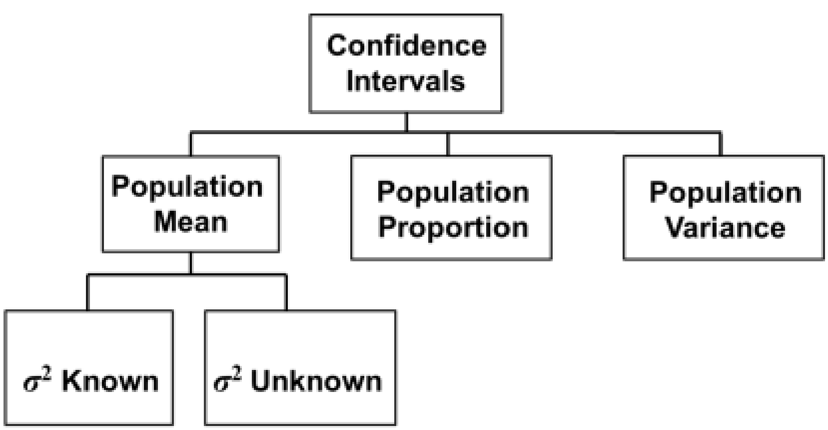
Interpretation (1 of 2)

- We are 95% confident that the true mean resistance is between 1.9932 and 2.4068 ohms
- Although the true mean may or may not be in this interval, 95% of intervals formed in this manner will contain the true mean

Section 7.3 Confidence Interval Estimation for the Mean (Sigma Squared Unknown)



(From normally distributed populations)

Student’s *t* Distribution (1 of 3)

- Consider a random sample of *n* observations
  - with mean $\overline{x}$ and standard deviation *s*
  - from a normally distributed population with mean μ
- Then the variable

$$t = \frac{\overline{x} - \mu}{\frac{s}{\sqrt{n}}}$$

follows the Student’s *t* distribution with (*n* − 1) degrees of freedom

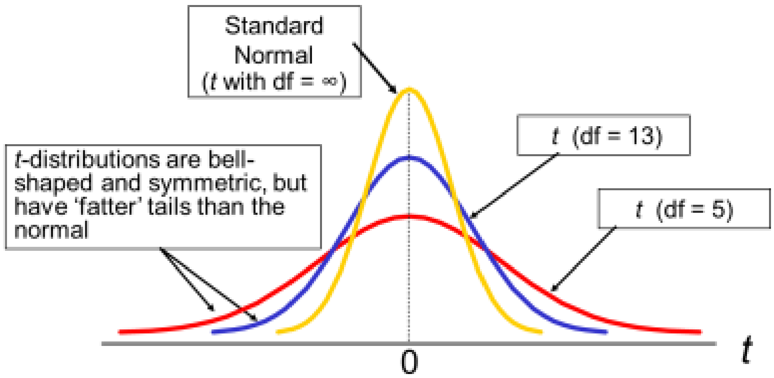
Student’s *t* Distribution (2 of 3)

- The *t* is a family of distributions
- The *t* value depends on degrees of freedom (d.f.)
  - Number of observations that are free to vary after sample mean has been calculated

$$\text{d.f.} = n - 1$$

Student's *t* Distribution (3 of 3)

Note: $t \rightarrow Z$ as *n* increases

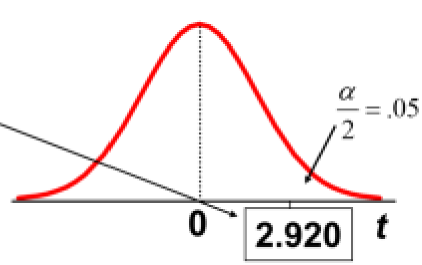


Student's *t* Table

df	Upper Tail Area .10	.05	.025
1	3.078	6.314	12.706
2	1.886	**2.920**	4.303
3	1.638	2.353	3.182

The body of the table contains *t* values, not probabilities

Let: $n = 3$
df = $n - 1 = 2$
$\alpha = .10$
$\frac{\alpha}{2} = .05$



t Distribution Values

With comparison to the *Z* value

Confidence Level	*t* (10 d.f.)	*t* (20 d.f.)	*t* (30 d.f.)	*z*
.80	1.372	1.325	1.310	1.282
.90	1.812	1.725	1.697	1.645
.95	2.228	2.086	2.042	1.960
.99	3.169	2.845	2.750	2.576

Note: $t \rightarrow Z$ as *n* increases

Confidence Interval Estimation for the Mean (Sigma Squared Unknown) (1 of 2)

- If the population standard deviation σ is unknown, we can substitute the sample standard deviation, *s*
- This introduces extra uncertainty, since *s* is variable from sample to sample
- So we use the *t* distribution instead of the normal distribution

Confidence Interval Estimation for the Mean (Sigma Squared Unknown) (2 of 2)

- Assumptions
  - Population standard deviation is unknown
  - Population is normally distributed
  - If population is not normal, use large sample
- Use Student's *t* Distribution
- Confidence Interval Estimate:

$$\bar{x} \pm t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

where $t_{n-1,\frac{\alpha}{2}}$ is the critical value of the *t* distribution with *n* − 1 d.f. and an area of $\frac{\alpha}{2}$ in each tail: $P\left(t_{n-1} > t_{n-1,\frac{\alpha}{2}}\right) = \frac{\alpha}{2}$

Margin of Error (2 of 2)

- The confidence interval,

$$\bar{x} \pm t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

- Can also be written as $\bar{x} \pm ME$

  where *ME* is called the margin of error:

$$ME = t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

Example 2

A random sample of n = 25 has $\overline{x} = 50$ and s = 8. Form a 95% confidence interval for μ

- d.f. = n − 1 = 24, so $t_{n-1,\frac{\alpha}{2}} = t_{24,.025} = 2.0639$

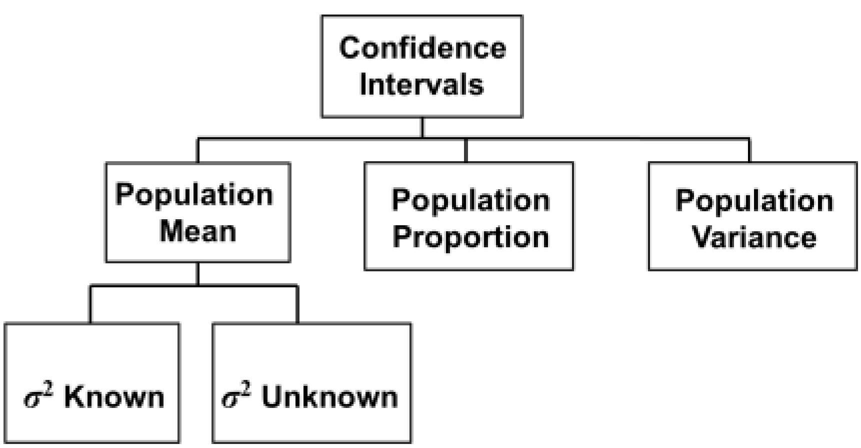
The confidence interval is

$$\overline{x} \pm t_{n-1,\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$$

$$50 \pm (2.0639) \frac{8}{\sqrt{25}}$$

$$46.698 < \mu < 53.302$$

Section 7.4 Confidence Interval Estimation for Population Proportion (1 of 2)



Section 7.4 Confidence Interval Estimation for Population Proportion (2 of 2)

- An interval estimate for the population proportion (P) can be calculated by adding an allowance for uncertainty to the sample proportion ($\hat{p}$)

Confidence Intervals for the Population Proportion

- Recall that the distribution of the sample proportion is approximately normal if the sample size is large, with standard deviation

$$\sigma_P = \sqrt{\frac{P(1-P)}{n}}$$

- We will estimate this with sample data:

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Confidence Interval Endpoints

- The confidence interval for the population proportion is given by

$$\hat{p} + z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- where
  - $z_{\frac{\alpha}{2}}$ is the standard normal value for the level of confidence desired
  - $\hat{p}$ is the sample proportion
  - n is the sample size
  - $nP(1-P) > 5$

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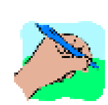
Example 3 (1 of 2)

- A random sample of 100 people shows that 25 are left-handed.
- Form a 95% confidence interval for the true proportion of left-handers



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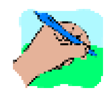
Example 3 (2 of 2)

- A random sample of 100 people shows that 25 are left-handed. Form a 95% confidence interval for the true proportion of left-handers.

$$\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

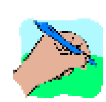
$$\frac{25}{100} \pm 1.96 \sqrt{\frac{.25(.75)}{100}}$$

$$0.1651 < P < 0.3349$$

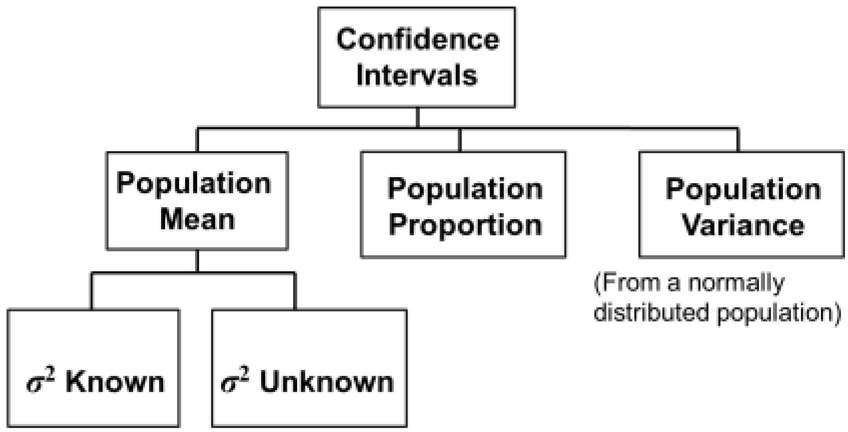


Interpretation (2 of 2)

- We are 95% confident that the true proportion of left-handers in the population is between 16.51% and 33.49%.
- Although the interval from 0.1651 to 0.3349 may or may not contain the true proportion, 95% of intervals formed from samples of size 100 in this manner will contain the true proportion.



Section 7.5 Confidence Interval Estimation for the Variance



Confidence Intervals for the Population Variance (1 of 3)

- Goal: Form a confidence interval for the population variance, σ^2
  - The confidence interval is based on the sample variance, s^2
  - Assumed: the population is normally distributed

Confidence Intervals for the Population Variance (2 of 3)

The random variable

$$\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma^2}$$

follows a chi-square distribution with (n − 1) degrees of freedom

Where the chi-square value $\chi^2_{n-1,\alpha}$ denotes the number for which

$$P\left(\chi^2_{n-1} > \chi^2_{n-1,\alpha}\right) = \alpha$$

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Confidence Intervals for the Population Variance (3 of 3)

The $100(1-\alpha)\%$ confidence interval for the population variance is given by

$$\text{LCL} = \frac{(n-1)s^2}{\chi^2_{n-1,\frac{\alpha}{2}}}$$

$$\text{UCL} = \frac{(n-1)s^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}$$

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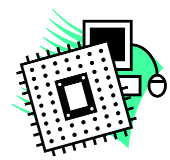
Example 4

You are testing the speed of a batch of computer processors. You collect the following data (in Mhz):

Sample size	**17**
Sample mean	**3004**
Sample std dev	**74**



Assume the population is normal.

Determine the 95% confidence interval for σ_x^2

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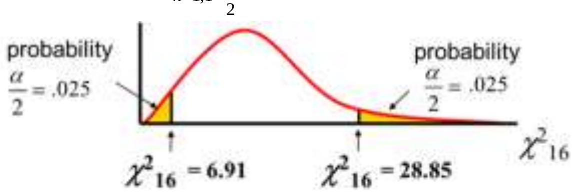
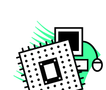
Finding the Chi-Square Values

- n = 17 so the chi-square distribution has (n − 1) = 16 degrees of freedom
- $\alpha = 0.05,$ so use the the chi-square values with area 0.025 in each tail:

$$\chi^2_{n-1,\frac{\alpha}{2}} = \chi^2_{16,0.025} = 28.85$$

$$\chi^2_{n-1,1-\frac{\alpha}{2}} = \chi^2_{16,0.975} = 6.91$$





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Calculating the Confidence Limits

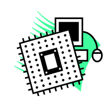
- The 95% confidence interval is

$$\frac{(n-1)s^2}{\chi^2_{n-1,\frac{\alpha}{2}}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}$$

$$\frac{(17-1)(74)^2}{28.85} < \sigma^2 < \frac{(17-1)(74)^2}{6.91}$$

$$3037 < \sigma^2 < 12680$$

Converting to standard deviation, we are 95% confident that the population standard deviation of CPU speed is between 55.1 and 112.6 Mhz



Section 7.6 Confidence Interval Estimation: Finite Populations

- If the sample size is more than 5% of the population size (and sampling is without replacement) then a finite population correction factor must be used when calculating the standard error

Finite Population Correction Factor

- Suppose sampling is without replacement and the sample size is large relative to the population size
- Assume the population size is large enough to apply the central limit theorem
- Apply the finite population correction factor when estimating the population variance

$$\text{finite population correction factor} = \frac{N-n}{N-1}$$

Estimating the Population Mean

- Let a simple random sample of size *n* be taken from a population of *N* members with mean μ
- The sample mean is an unbiased estimator of the population mean μ
- The point estimate is:

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Finite Populations: Mean

- If the sample size is more than 5% of the population size, an unbiased estimator for the variance of the sample mean is

$$\hat{\sigma}_{\bar{x}}^2 = \frac{s^2}{n}\left(\frac{N-n}{N-1}\right)$$

- So the $100(1-\alpha)\%$ confidence interval for the population mean is

$$\bar{x} \pm t_{n-1,\frac{\alpha}{2}}\hat{\sigma}_{\bar{x}}$$

Estimating the Population Total (1 of 2)

- Consider a simple random sample of size *n* from a population of size *N*
- The quantity to be estimated is the population total $N\mu$
- An unbiased estimation procedure for the population total $N\mu$ yields the point estimate $N\bar{x}$

Estimating the Population Total (2 of 2)

- An unbiased estimator of the variance of the population total is

$$N^2 \hat{\sigma}_{\bar{x}}^2 = N^2 \frac{s^2}{n} \left(\frac{N-n}{N-1} \right)$$

- A $100(1-\alpha)\%$ confidence interval for the population total is

$$N\bar{x} \pm t_{n-1,\frac{\alpha}{2}} N\hat{\sigma}_{\bar{x}}$$

Confidence Interval for Population Total: Example

A firm has a population of 1000 accounts and wishes to estimate the value of the total population balance

A sample of 80 accounts is selected with average balance of $87.60 and standard deviation of $22.30

Find the 95% confidence interval estimate of the total balance

Example Solution

$$N = 1000, \quad n = 80, \quad \bar{x} = 87.6, \quad s = 22.3$$

$$N^2 \hat{\sigma}_{\bar{x}}^2 = N^2 \frac{s^2}{n} \frac{(N-n)}{N-1} = (1000)^2 \frac{(22.3)^2}{80} \frac{920}{999} = 5724559.6$$

$$N \hat{\sigma}_{\bar{x}} = \sqrt{5724559.6} = 2392.6$$

$$N\bar{x} \pm t_{79,0.025} N \hat{\sigma}_{\bar{x}} = (1000)(87.6) \pm (1.9905)(2392.6)$$

$$82837.53 < N\mu < 92362.47$$

The 95% confidence interval for the population total balance is $82,837.53 to $92,362.47

Estimating the Population Proportion: Finite Population

- Let the true population proportion be *P*
- Let $\hat{p}$ be the sample proportion from *n* observations from a simple random sample
- The sample proportion, $\hat{p}$, is an unbiased estimator of the population proportion, *P*

Confidence Intervals for Population Proportion: Finite Population

- If the sample size is more than 5% of the population size, an unbiased estimator for the variance of the population proportion is

$$\hat{\sigma}_{\hat{p}}^{2} = \frac{\hat{p}(1-\hat{p})}{n}\left(\frac{N-n}{N-1}\right)$$

- So the $100(1-\alpha)\%$ confidence interval for the population proportion is

$$\hat{p} \pm z_{\frac{\alpha}{2}}\hat{\sigma}_{\hat{p}}$$

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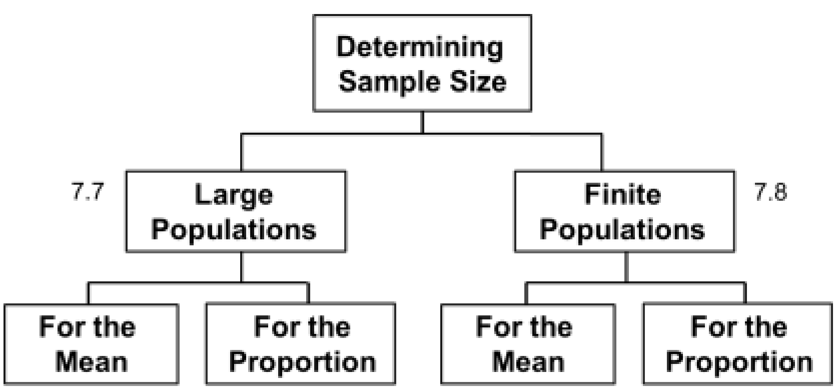
Sample-Size Determination (1 of 2)



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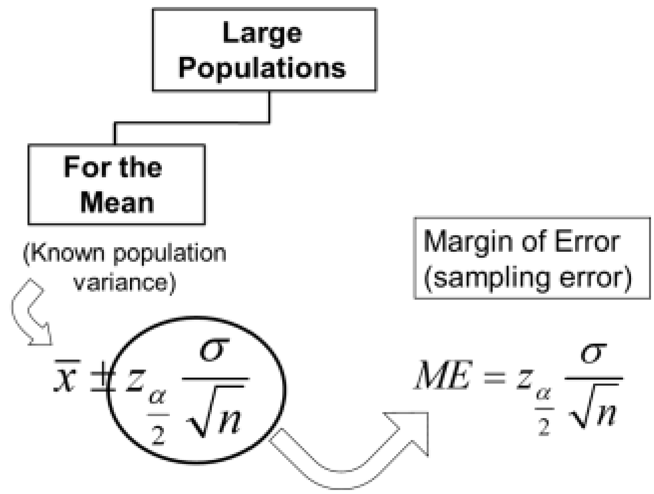
Section 7.7 Sample-Size Determination: Large Populations (1 of 2)



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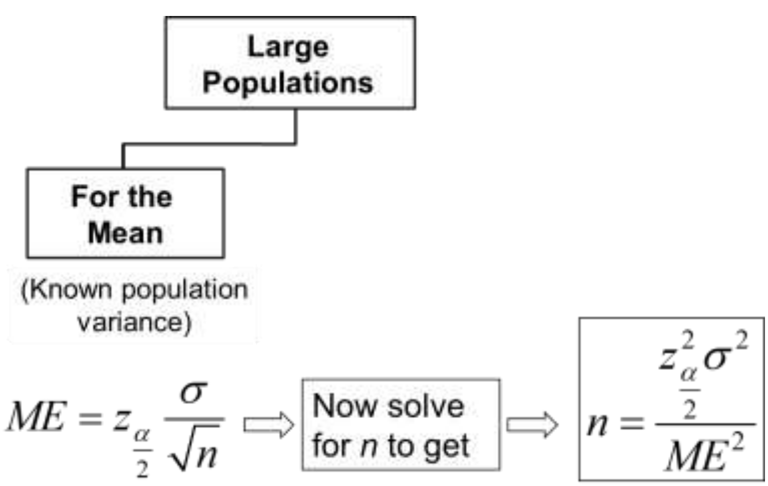
Section 7.7 Sample-Size Determination: Large Populations (2 of 2)



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Sample-Size Determination (2 of 2)

- To determine the required sample size for the mean, you must know:
  - The desired level of confidence $(1-\alpha)$, which determines the $z_{\frac{\alpha}{2}}$ value
  - The acceptable margin of error (sampling error), *ME*
  - The population standard deviation, σ

Required Sample Size Example

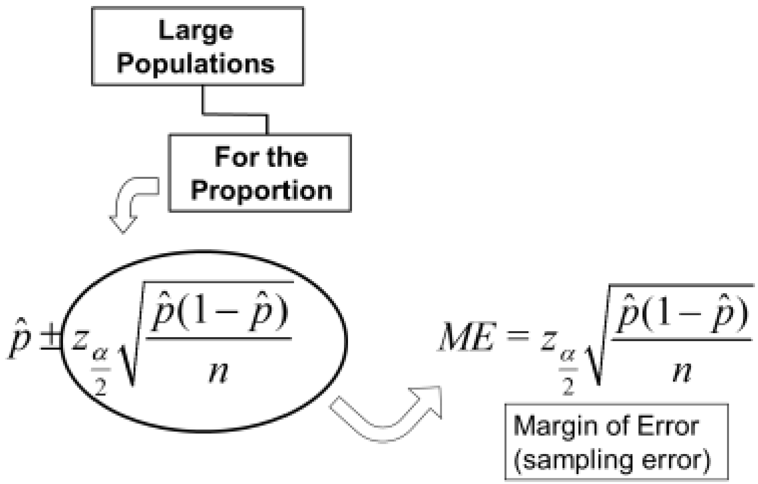
If $\sigma = 45$, what sample size is needed to estimate the mean within ± 5 with 90% confidence?

$$n = \frac{z_{\frac{\alpha}{2}}^2 \sigma^2}{ME^2} = \frac{(1.645)^2 (45)^2}{5^2} = 219.19$$

So the required sample size is **n = 220**

(Always round up)

Sample Size Determination: Population Proportion (1 of 3)



Pearson

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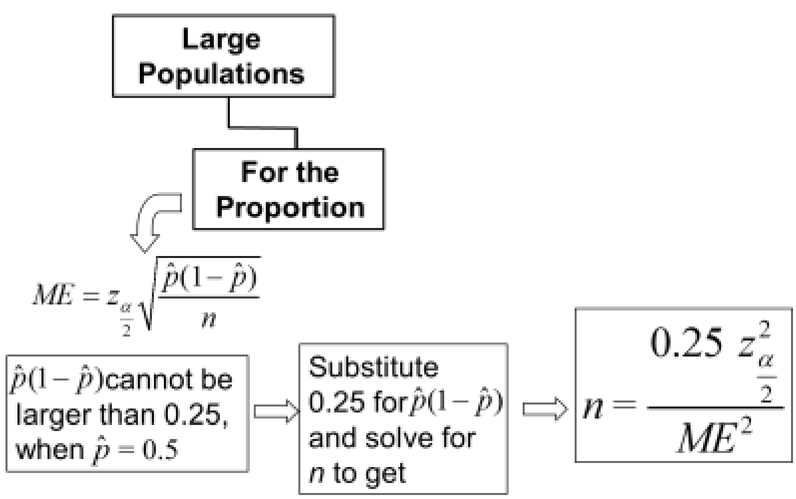
Sample Size Determination: Population Proportion (2 of 3)



Pearson

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Sample Size Determination: Population Proportion (3 of 3)

- The sample and population proportions, $\hat{p}$ and *P*, are generally not known (since no sample has been taken yet)
- $P(1-P)=0.25$ generates the largest possible margin of error (so guarantees that the resulting sample size will meet the desired level of confidence)
- To determine the required sample size for the proportion, you must know:
  - The desired level of confidence $(1-\alpha)$, which determines the critical $z_{\frac{\alpha}{2}}$ value
  - The acceptable sampling error (margin of error), *ME*
  - Estimate $P(1-P)=0.25$

Required Sample Size Example: Population Proportion (1 of 2)

How large a sample would be necessary to estimate the true proportion defective in a large population within $\pm 3\%$, with 95% confidence?

Required Sample Size Example: Population Proportion (2 of 2)

Solution:

For 95% confidence, use $z_{0.025} = 1.96$

ME = 0.03

Estimate $P(1-P) = 0.25$

$$n = \frac{0.25\ z_{\frac{\alpha}{2}}^2}{ME^2} = \frac{(0.25)(1.96)^2}{(0.03)^2} = 1067.11$$

So use *n* = 1068

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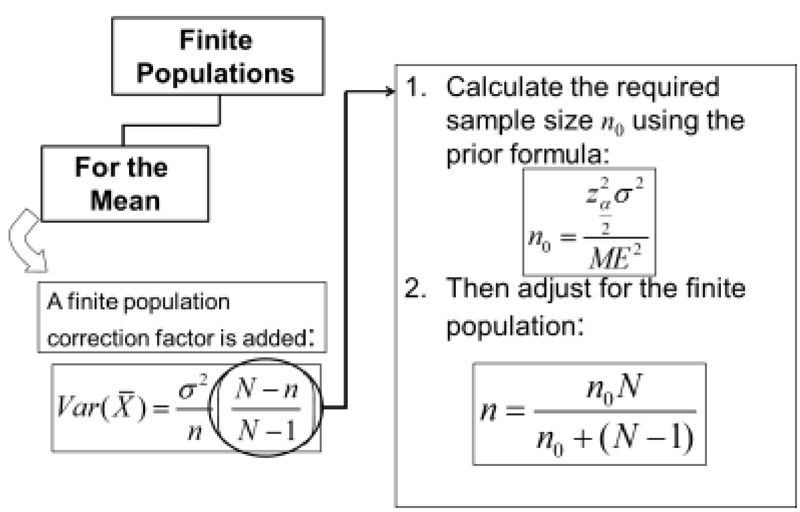
Section 7.8 Sample-Size Determination: Finite Populations (1 of 2)



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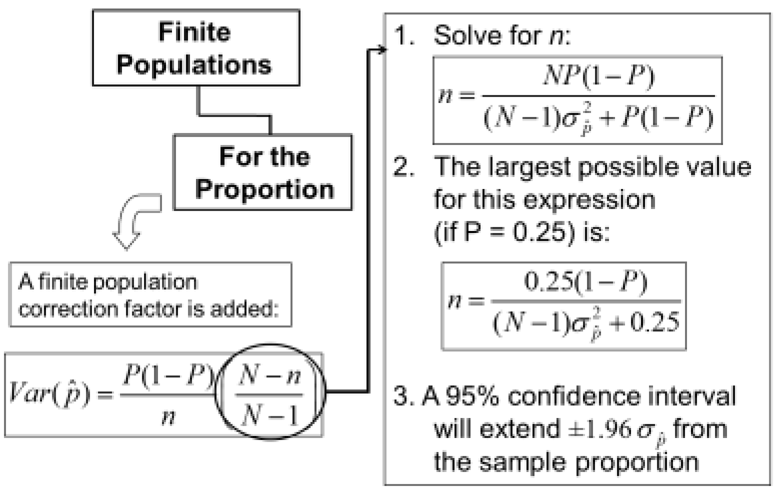
Section 7.8 Sample-Size Determination: Finite Populations (2 of 2)



Example 5: Sample Size to Estimate Population Proportion (1 of 2)

How large a sample would be necessary to estimate within $\pm 5\%$ the true proportion of college graduates college graduates in a population of 850 people with 95% confidence?

Example 5: Sample Size to Estimate Population Proportion (2 of 2)

Solution:

- For 95% confidence, use $z_{0.025} = 1.96$
- *ME* = 0.05

$$1.96\ \sigma_{\hat{p}} = 0.05 \Rightarrow \sigma_{\hat{p}} = 0.02551$$

$$n_{\max} = \frac{0.25N}{(N-1)\sigma_{\hat{p}}^2 + 0.25} = \frac{(0.25)(850)}{(849)(0.02551)^2 + 0.25} = 264.8$$

So use *n* = 265

Chapter Summary (1 of 2)

- Introduced the concept of confidence intervals
- Discussed point estimates
- Developed confidence interval estimates
- Created confidence interval estimates for the mean (σ^2 known)
- Introduced the Student's *t* distribution
- Determined confidence interval estimates for the mean (σ^2 unknown)

Chapter Summary (2 of 2)

- Created confidence interval estimates for the proportion
- Created confidence interval estimates for the variance of a normal population
- Applied the finite population correction factor to form confidence intervals when the sample size is not small relative to the population size
- Determined required sample size to meet confidence and margin of error requirements