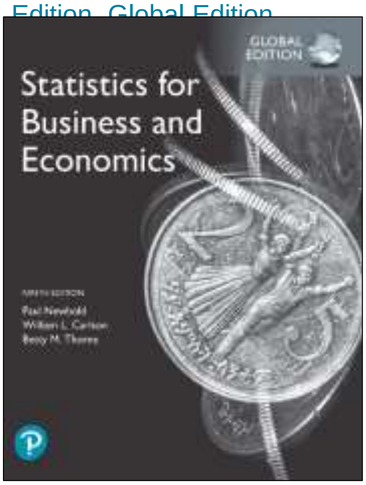
Statistics for Business and Economics

Ninth Edition, Global Edition



Chapter 9

Hypothesis Testing: Single Population

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Chapter Goals

After completing this chapter, you should be able to:

- Formulate null and alternative hypotheses for applications involving
  - a single population mean from a normal distribution
  - a single population proportion (large samples)
  - the variance of a normal distribution
- Formulate a decision rule for testing a hypothesis
- Know how to use the critical value and *p*-value approaches to test the null hypothesis (for both mean and proportion problems)
- Define Type I and Type II errors and assess the power of a test
- Use the chi-square distribution for tests of the variance of a normal distribution

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Section 9.1 Concepts of Hypothesis Testing

- A hypothesis is a claim (assumption) about a population parameter:
  - population mean

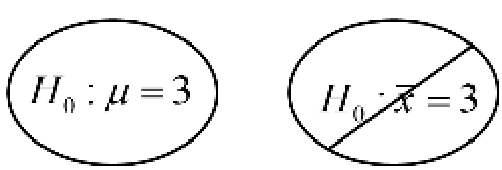
    Example: The mean monthly cell phone bill of this city is $\mu = \$52$
  - population proportion

    Example: The proportion of adults in this city with cell phones is *P* = .88

The Null Hypothesis, *H* Sub 0 (1 of 2)

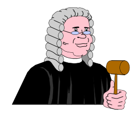
- States the assumption (numerical) to be tested

  Example: The average number of TV sets in U.S. Homes is equal to three $(H_0 : \mu = 3)$
- Is always about a population parameter, not about a sample statistic



The Null Hypothesis, *H* sub 0 (2 of 2)

- Begin with the assumption that the null hypothesis is true
  - Similar to the notion of innocent until proven guilty



- Refers to the status quo
- Always contains "=", "≤" or "≥" sign
- May or may not be rejected

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The Alternative Hypothesis, *H* Sub 1

- Is the opposite of the null hypothesis
  - e.g., The average number of TV sets in U.S. homes is not equal to 3 $(H_1 : \mu \neq 3)$
- Challenges the status quo
- Never contains the "=", "≤" or "≥" sign
- May or may not be supported
- Is generally the hypothesis that the researcher is trying to support

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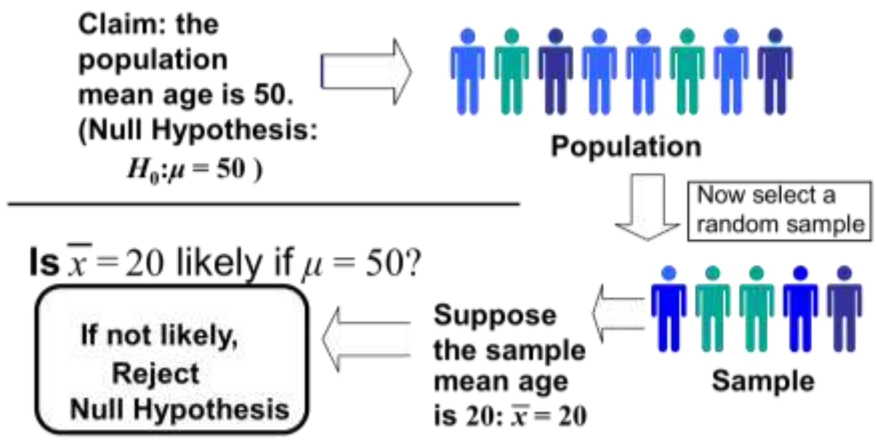
Hypothesis Testing Process



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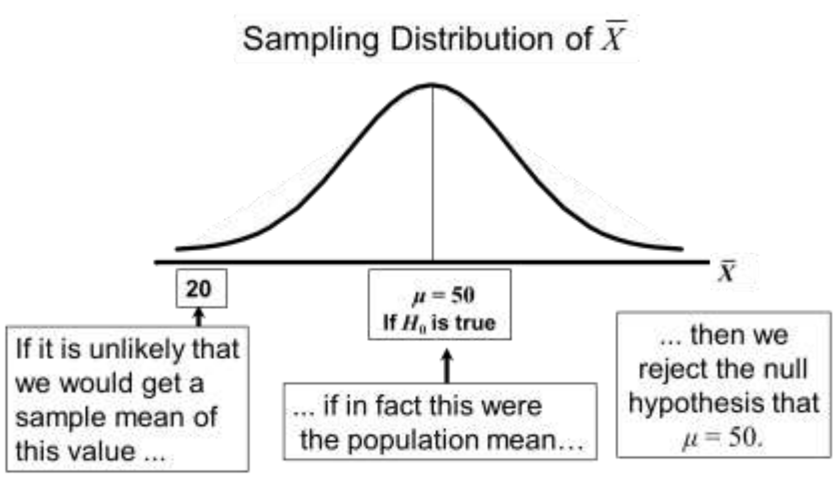
Reason for Rejecting *H* Sub 0



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Level of Significance, α

- **Defines the unlikely values of the sample statistic if the null hypothesis is true**
  - Defines rejection region of the sampling distribution
- Is designated by $\boldsymbol{\alpha}$, (level of significance)
  - Typical values are 0.01, 0.05, or 0.10
- Is selected by the researcher at the beginning
- Provides the critical value(s) of the test

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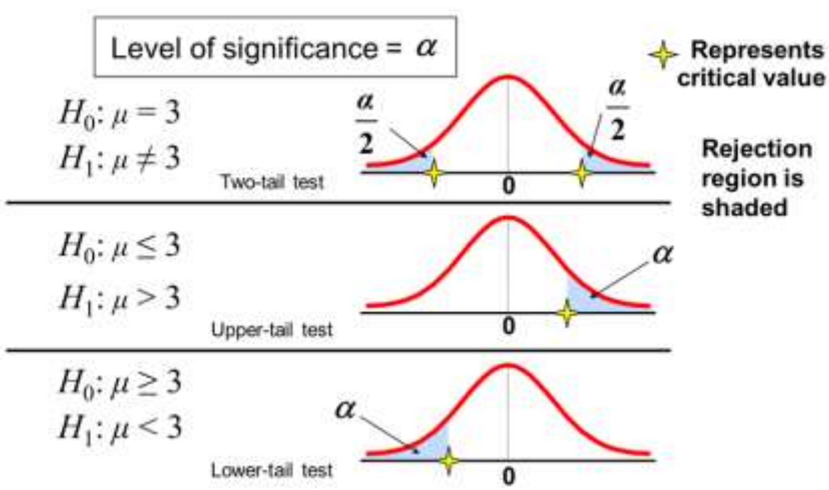
Level of Significance and the Rejection Region



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Errors in Making Decisions (1 of 2)

- **Type I Error**
  - Reject a true null hypothesis
  - Considered a serious type of error

The probability of Type I Error is α

- Called level of significance of the test
- Set by researcher in advance

Errors in Making Decisions (2 of 2)

- **Type II Error**
  - Fail to reject a false null hypothesis

The probability of Type II Error is β

Outcomes and Probabilities

Possible Hypothesis Test Outcomes

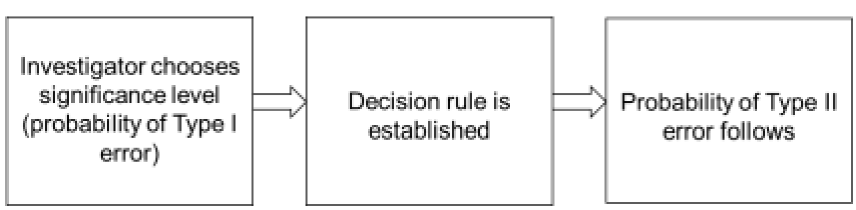
	Actual Situation	
Decision	H_0 True	H_0 False
Fail to Reject H_0	Correct Decision $(1-\alpha)$	Type II Error (β)
Reject H_0	Type I Error (α)	Correct Decision $(1-\beta)$

Key:
Outcome
(Probability)

$(1-\beta)$ is called the power of the test

Consequences of Fixing the Significance Level of a Test

- Once the significance level α is chosen (generally less than 0.10), the probability of Type II error, β, can be found.



Type I & II Error Relationship

- Type I and Type II errors can not happen at the same time
  - Type I error can only occur if H_0 is true
  - Type II error can only occur if H_0 is false

If Type I error probability $(\alpha) \Uparrow$, then

Type II error probability $(\beta) \Downarrow$

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Factors Affecting Type II Error

- All else equal,
  - $\beta \Uparrow$ when the difference between hypothesized parameter and its true value $\Downarrow$
  - $\beta \Uparrow$ when $\alpha \Downarrow$
  - $\beta \Uparrow$ when $\sigma \Uparrow$
  - $\beta \Uparrow$ when $n \Downarrow$

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Power of the Test

- The power of a test is the probability of rejecting a null hypothesis that is false
- i.e., $\text{Power} = P(\text{Reject } H_0 \mid H_1 \text{ is true})$
  - Power of the test increases as the sample size increases

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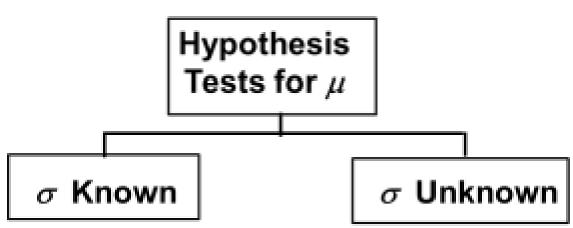
Hypothesis Tests for the Mean



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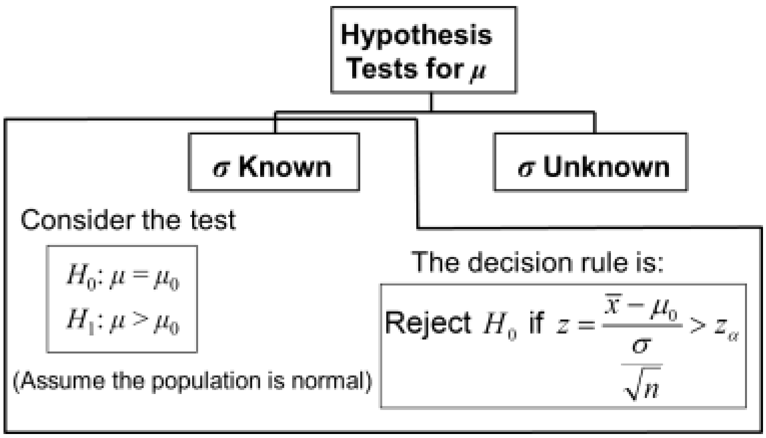
Section 9.2 Tests of the Mean of a Normal Distribution Sigma Known

- Convert sample result $(\bar{x})$ to a z value

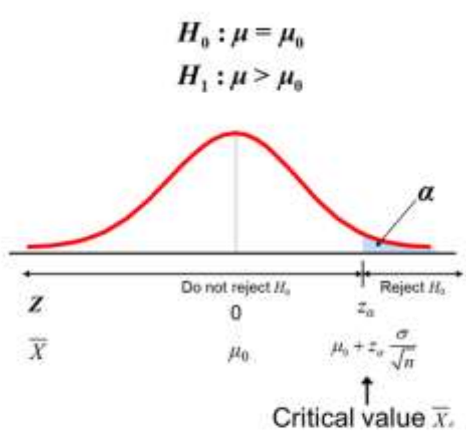


Decision Rule

Reject H_0 if $z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} > z_\alpha$

Alternate rule:

Reject H_0 if $\bar{x} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$



p-Value

- *p*-value: Probability of obtaining a test statistic more extreme $(\leq \text{ or } \geq)$ than the observed sample value given H_0 is true
  - Also called observed level of significance
  - Smallest value of ***α*** for which H_0 can be rejected

p-Value Approach to Testing

- Convert sample result (e.g., $\bar{x}$) to test statistic (e.g., *z* statistic)
- Obtain the *p*-value
  - For an upper tail test:

$$p\text{-value} = P\left(z > \frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}}, \text{ given thet } H_0 \text{ is true}\right)$$

$$= P\left(z > \frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}} \middle| \mu = \mu_0\right)$$

- Decision rule: compare the *p*-value to ***α***
  - If *p*-value $< \boldsymbol{\alpha}$, reject H_0
  - If *p*-value $\geq \boldsymbol{\alpha}$, do not reject H_0

Example 1: Upper-Tail *Z* Test for Mean Sigma Known

A phone industry manager thinks that customer monthly cell phone bill have increased, and now average over $52 per month. The company wishes to test this claim. (Assume $\sigma = 10$ is known)

Form hypothesis test:

$H_0: \mu \le 52$ the average is not over $52 per month

$H_1: \mu > 52$ the average is greater than $52 per month (i.e., sufficient evidence exists to support the manager's claim)

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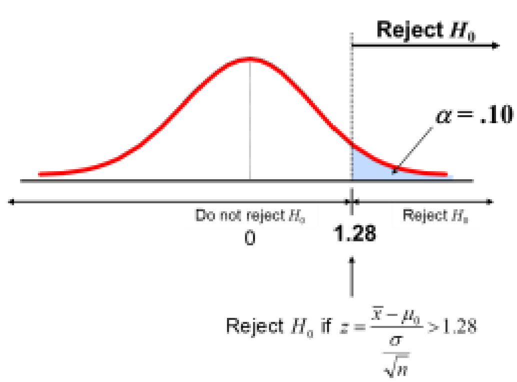
Example 2: Find Rejection Region

- Suppose that $\alpha = .10$ is chosen for this test

Find the rejection region:



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Example 3: Sample Results

Obtain sample and compute the test statistic

Suppose a sample is taken with the following results: $n = 64,\ \bar{x} = 53.1$ ($\sigma = 10$ was assumed known)

- Using the sample results,

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{53.1 - 52}{\frac{10}{\sqrt{64}}} = 0.88$$

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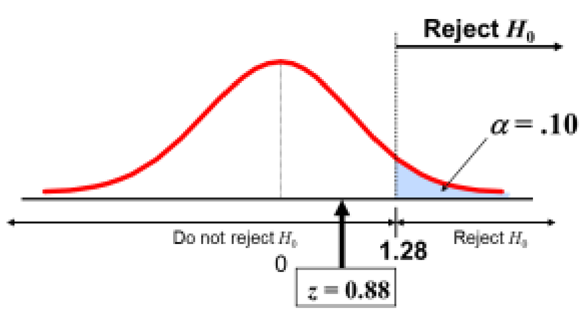
Example 4: Decision

Reach a decision and interpret the result:



Do not reject H_0 since $z = 0.88 < 1.28$

i.e.: there is not sufficient evidence that the mean bill is over $52

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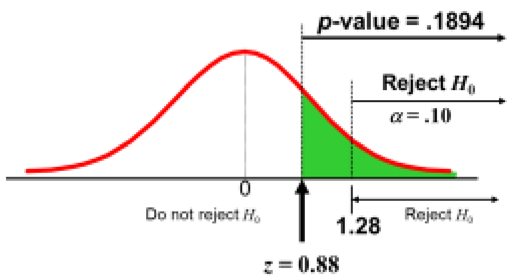
Example 5: *p*-Value Solution

Calculate the *p*-value and compare to ***α***
(assuming that $\mu = 52.0$)



$$P(\bar{x} \geq 53.1 \mid \mu = 52.0)$$
$$= P\left(z \geq \frac{53.1 - 52.0}{\frac{10}{\sqrt{64}}}\right)$$
$$= P(z \geq 0.88) = 1 - .8106$$
$$= .1894$$

Do not reject H_0 since *p*-value $= .1894 > \alpha = .10$

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One-Tail Tests

- In many cases, the alternative hypothesis focuses on one particular direction

$H_0: \mu \leq 3$
$H_1: \mu > 3$
⟹ This is an upper-tail test since the alternative hypothesis is focused on the upper tail above the mean of 3

$H_0: \mu \geq 3$
$H_1: \mu < 3$
⟹ This is a lower-tail test since the alternative hypothesis is focused on the lower tail below the mean of 3

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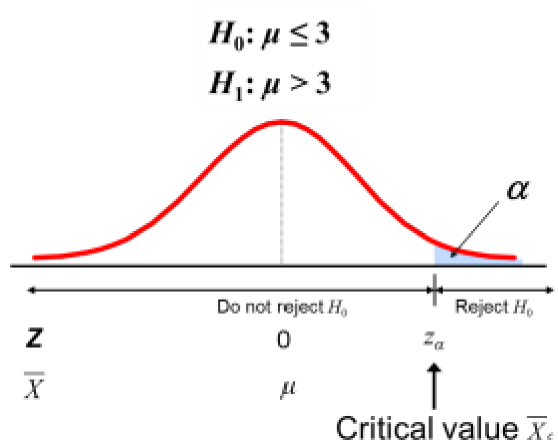
Upper-Tail Tests

- There is only one critical value, since the rejection area is in only one tail



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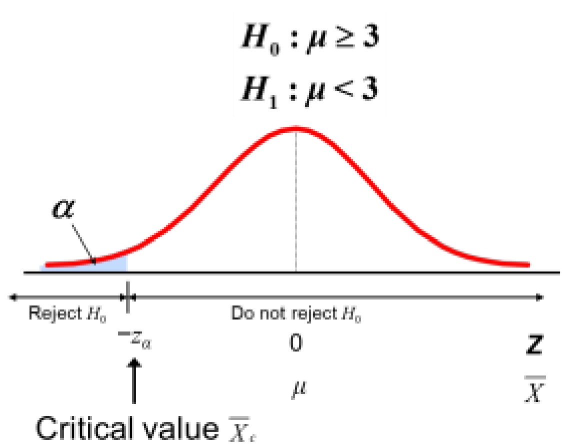
Lower-Tail Tests

- There is only one critical value, since the rejection area is in only one tail



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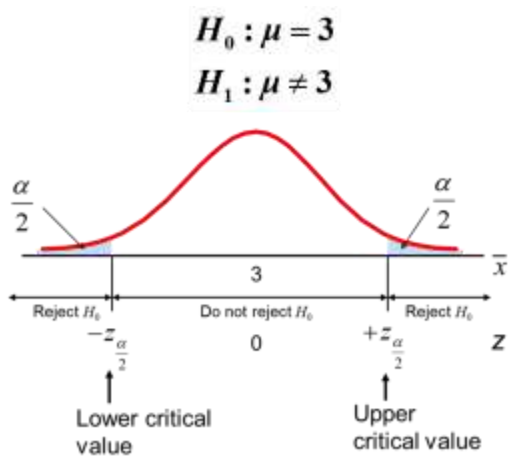
Two-Tail Tests

- In some settings, the alternative hypothesis does not specify a unique direction
- There are two critical values, defining the two regions of rejection



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Hypothesis Testing Example (1 of 4)

Test the claim that the true mean # of TV sets in US homes is equal to 3. (Assume $\sigma = 0.8$)

- State the appropriate null and alternative hypotheses
  - $H_0: \mu = 3,\ H_1: \mu \neq 3$ (This is a two tailed test)
- Specify the desired level of significance
  - Suppose that $\alpha = .05$ is chosen for this test
- Choose a sample size
  - Suppose a sample of size *n* = 100 is selected

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Hypothesis Testing Example (2 of 4)

- Determine the appropriate technique
  - σ is known so this is a *z* test
- Set up the critical values
  - For $\alpha = .05$ the critical *z* values are ± 1.96
- Collect the data and compute the test statistic
  - Suppose the sample results are $n = 100,\ \bar{x} = 2.84$ ($\sigma = 0.8$ is assumed known)

So the test statistic is:

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{2.84 - 3}{\frac{0.8}{\sqrt{100}}} = \frac{-.16}{.08} = -2.0$$

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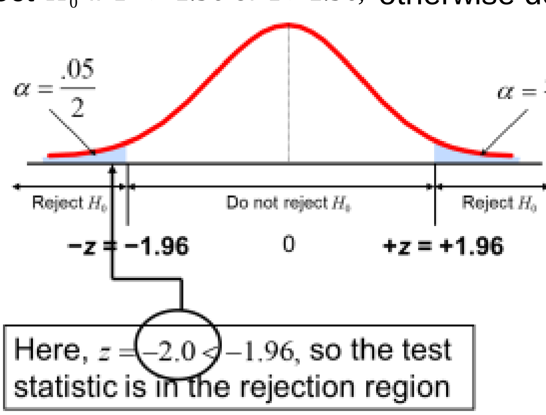
Hypothesis Testing Example (3 of 4)

- Is the test statistic in the rejection region?

Reject H_0 if $z < -1.96$ or $z > 1.96$; otherwise do not reject H_0



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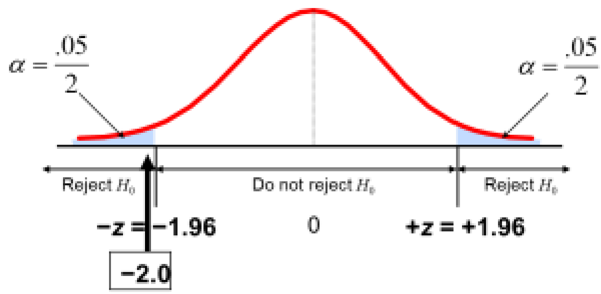
Hypothesis Testing Example (4 of 4)

- Reach a decision and interpret the result



Since $z = -2.0 < -1.96$, we reject the null hypothesis and conclude that there is sufficient evidence that the mean number of TVs in US homes is not equal to 3

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Example 6: *p*-Value (1 of 2)

- Example: How likely is it to see a sample mean of 2.84 (or something further from the mean, in either direction) if the true mean is $\mu = 3.0$?

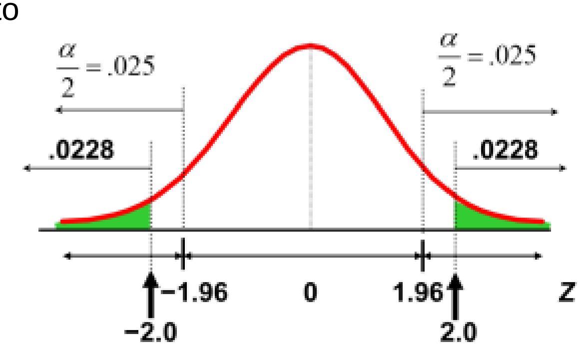
$\bar{x} = 2.84$ is translated to a *z* score of *z* = −2.0

$P(z < -2.0) = .0228$

$P(z > 2.0) = .0228$

***p*-value**

$= .0228 + .0228 = .0456$



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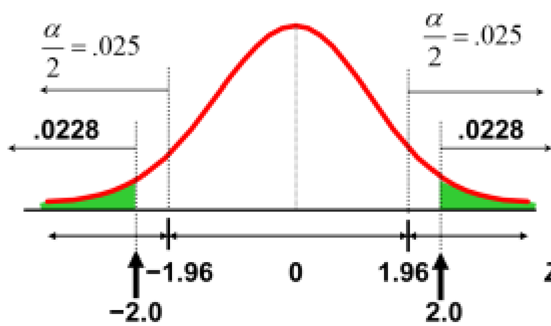
Example 6: *p*-Value (2 of 2)

- Compare the *p*-value to α
  - If $p\text{-value} < \alpha$, reject H_0
  - If $p\text{-value} \geq \alpha$, do not reject H_0

Here: $p\text{-value} = .0456$

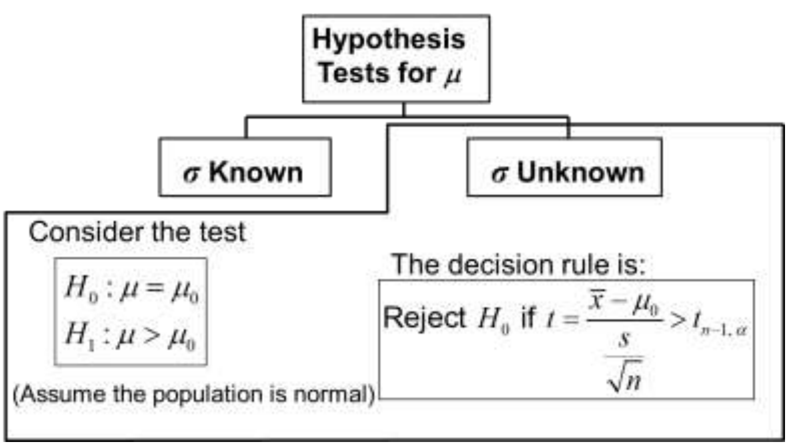
$\alpha = .05$

Since .0456 < .05, we reject the null hypothesis



Section 9.3 Tests of the Mean of a Normal Population Sigma Unknown (1 of 2)

- Convert sample result $(\bar{x})$ to a *t* test statistic



Section 9.3 Tests of the Mean of a Normal Population sigma unknown (2 of 2)

- For a two-tailed test:

Consider the test

$H_0 : \mu = \mu_0$
$H_1 : \mu \neq \mu_0$

(Assume the population is normal, and the population variance is unknown)

The decision rule is:

Reject H_0 if $t = \dfrac{\bar{x} - \mu_0}{\dfrac{s}{\sqrt{n}}} < -t_{n-1,\frac{\alpha}{2}}$ or if $t = \dfrac{\bar{x} - \mu_0}{\dfrac{s}{\sqrt{n}}} > t_{n-1,\frac{\alpha}{2}}$

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Example 7: Two-Tail Test Sigma Unknown

The average cost of a hotel room in Chicago is said to be \$168 per night. A random sample of 25 hotels resulted in $\bar{x} = \$172.50$ and $s = \$15.40$. Test at the $\alpha = 0.05$ level.

(Assume the population distribution is normal)

$H_0 : \mu = 168$
$H_1 : \mu \neq 168$

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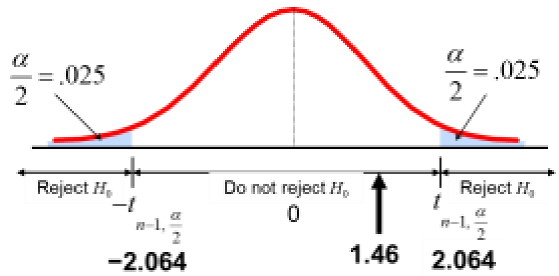
Example Solution: Two-Tail Test

$$H_0 : \mu = 168$$
$$H_1 : \mu \neq 168$$



- $\alpha = 0.05$
- $n = 25$
- σ **is unknown, so use a *t* statistic** →

$$t_{n-1} = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{172.50 - 168}{\frac{15.40}{\sqrt{25}}} = 1.46$$

- **Critical Value:**

$t_{24, .025} = \pm 2.064$

Do not reject H_0 **:** not sufficient evidence that true mean cost is different than $168

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Section 9.4 Tests of the Population Proportion (1 of 2)

- Involves categorical variables
- Two possible outcomes
  - "Success" (a certain characteristic is present)
  - "Failure" (the characteristic is not present)
- Fraction or proportion of the population in the "success" category is denoted by *P*
- Assume sample size is large

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Section 9.4 Tests of the Population Proportion (2 of 2)

- The sample proportion in the success category is denoted by $\hat{p}$
  - $\hat{p} = \frac{x}{n} = \frac{\text{number of successes in sample}}{\text{sample size}}$
- When $nP(1-P) > 5$, $\hat{p}$ can be approximated by a normal distribution with mean and standard deviation
  - $\mu_{\hat{p}} = P \qquad \sigma_{\hat{p}} = \sqrt{\frac{P(1-P)}{n}}$

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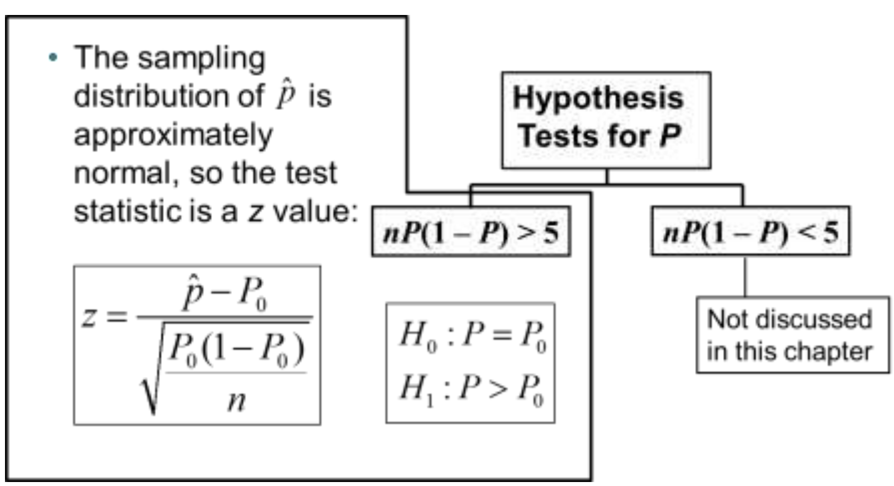
Hypothesis Tests for Proportions



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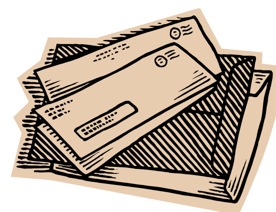
Example 8: *Z* Test for Proportion

A marketing company claims that it receives 8% responses from its mailing. To test this claim, a random sample of 500 were surveyed with 25 responses. Test at the $\alpha = .05$ significance level.



Check:

Our approximation for *P* is

$$\hat{p} = \frac{25}{500} = .05$$

$$nP(1-P) = (500)(.05)(.95) = 23.75 > 5$$ ✓

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Z Test for Proportion: Solution

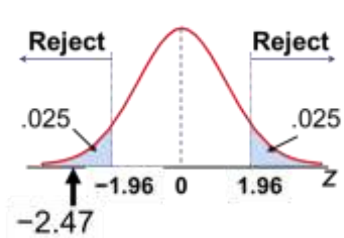
$\boldsymbol{H_0 : P = .08}$

$\boldsymbol{H_1 : P \neq .08}$

$\alpha = .05$

$n = 500, \ \hat{p} = .05$

Critical Values: ±1.96



Test Statistic:

$$z = \frac{\hat{p} - P_0}{\sqrt{\frac{P_0(1-P_0)}{n}}} = \frac{.05 - .08}{\sqrt{\frac{.08(1-.08)}{500}}} = -2.47$$

Decision:

Reject H_0 at $\alpha = .05$

Conclusion:

There is sufficient evidence to reject the company's claim of 8% response rate.

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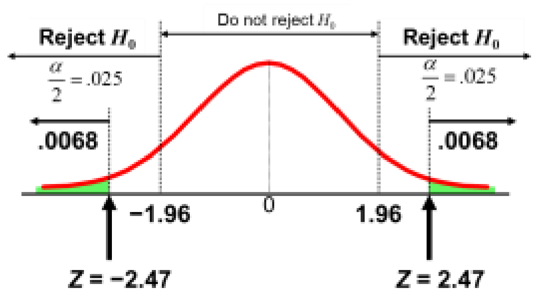
p-Value Solution

Calculate the *p*-value and compare to α

(For a two sided test the *p*-value is always two sided)



***p*-value = .0136:**

$P(Z \le -2.47) + P(Z \ge 2.47)$

$= 2(.0068) = 0.0136$

Reject H_0 since *p*-value $= .0136 < \alpha = .05$

Section 9.5 Assessing the Power of a Test

- Recall the possible hypothesis test outcomes:

Key: Outcome (Probability)

	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject H_0	Correct Decision ($1-\alpha$)	Type II Error (β)
Reject H_0	Type I Error (α)	Correct Decision ($1-\beta$)

- β denotes the probability of Type II Error
- $1-\beta$ is defined as the power of the test

Power $= 1-\beta$ = the probability that a false null hypothesis is rejected

Type II Error

Assume the population is normal and the population variance is known. Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0$$

The decision rule is:

Reject H_0 if $z = \dfrac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} > z_\alpha$ or Reject H_0 if $\bar{x} > \bar{x}_c = \mu_0 + z_\alpha \dfrac{\sigma}{\sqrt{n}}$

If the null hypothesis is false and the true mean is μ^*,
then the probability of type II error is

$$\beta = P(\bar{x} < \bar{x}_c \mid \mu = \mu^*) = P\left(z < \dfrac{\bar{x}_c - \mu^*}{\frac{\sigma}{\sqrt{n}}} \right)$$

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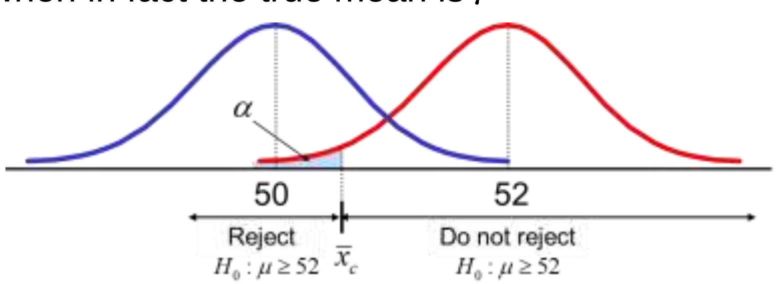
Type II Error Example (1 of 3)

- Type II error is the probability of failing to reject a false H_0

Suppose we fail to reject $\boldsymbol{H_0 : \mu \geq 52}$
when in fact the true mean is $\boldsymbol{\mu^* = 50}$



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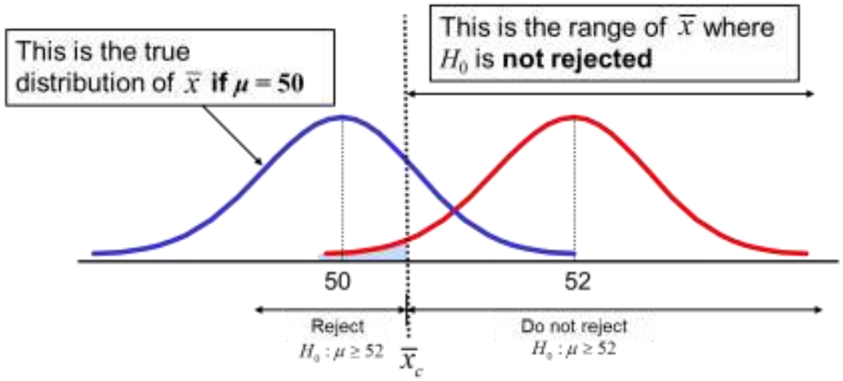
Type II Error Example (2 of 3)

- Suppose we do not reject $H_0: \mu \geq 52$ when in fact the true mean is $\mu^* = 50$



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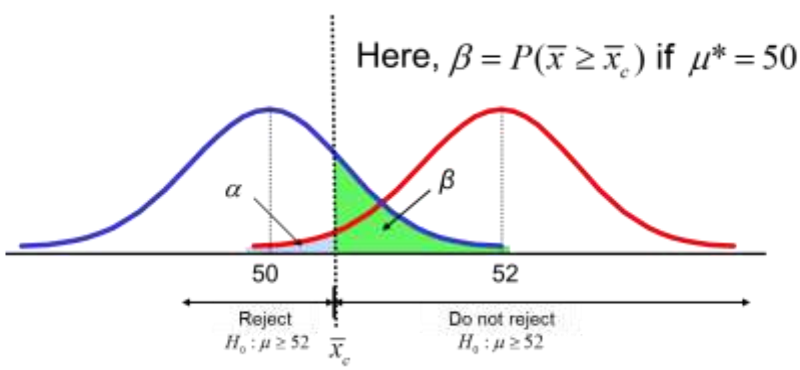
Type II Error Example (3 of 3)

- Suppose we do not reject $H_0: \mu \geq 52$ when in fact the true mean is $\mu^* = 50$



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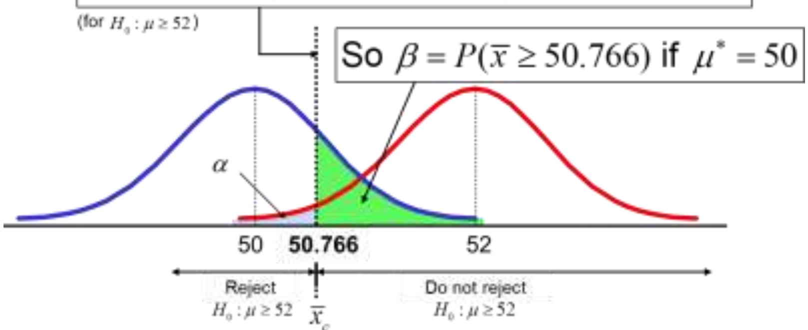
Calculating β (1 of 2)

- Suppose $n = 64$, $\sigma = 6$, and $\alpha = .05$

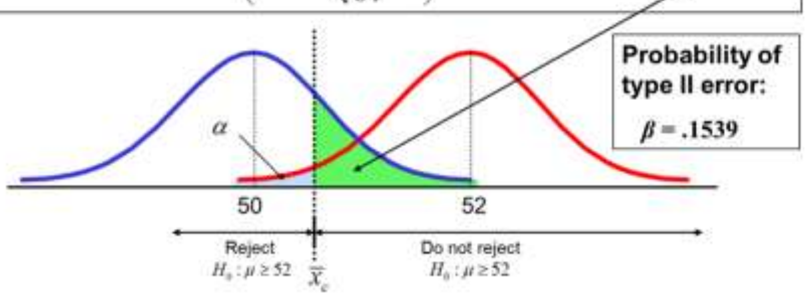
$$\bar{x}_c = \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} = 52 - 1.645 \frac{6}{\sqrt{64}} = \boxed{50.766}$$



Calculating β (2 of 2)

- Suppose $n = 64$, $\sigma = 6$, and $\alpha = .05$

$$P(\bar{x} \ge 50.766 \mid \mu^* = 50)\ P = \left(z \ge \frac{50.766 - 50}{\frac{6}{\sqrt{64}}} \right) = P(z \ge 1.02) = .5 - .3461 = \boxed{.1539}$$



Power of the Test Example

If the true mean is $\mu^* = 50$,

- The probability of Type II Error $= \beta = 0.1539$
- The power of the test $= 1 - \beta = 1 - 0.1539 = 0.8461$

Key: Outcome (Probability)

	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject H_0	Correct Decision $1 - \alpha = 0.95$	Type II Error $\beta = 0.1539$
Reject H_0	Type I Error $\alpha = 0.05$	Correct Decision $1 - \beta = 0.8461$

(The value of β and the power will be different for each μ^*)

Section 9.6 Tests of the Variance of a Normal Distribution (1 of 2)

- Goal: Test hypotheses about the population variance, σ^2 (e.g., $H_0 : \sigma^2 = \sigma_0^2$)
  - If the population is normally distributed,

$$\chi_{n-1}^2 = \frac{(n-1)s^2}{\sigma^2}$$

has a chi-square distribution with $(n-1)$ degrees of freedom

Section 9.6 Tests of the Variance of a Normal Distribution (2 of 2)

The test statistic for hypothesis tests about one population variance is

$$\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma_0^2}$$

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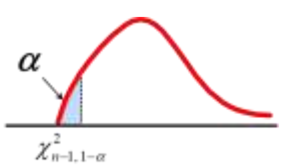
Decision Rules: Variance

Population variance

Lower-tail test:

$H_0: \sigma^2 \geq \sigma_0^2$

$H_1: \sigma^2 < \sigma_0^2$

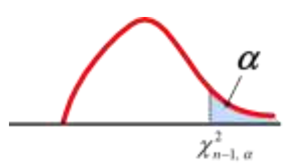


Reject H_0 if $\chi^2_{n-1} < \chi^2_{n-1,1-\alpha}$

Upper-tail test:

$H_0: \sigma^2 \leq \sigma_0^2$

$H_1: \sigma^2 > \sigma_0^2$

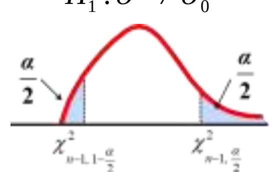


Reject H_0 if $\chi^2_{n-1} > \chi^2_{n-1,\alpha}$

Two-tail test:

$H_0: \sigma^2 = \sigma_0^2$

$H_1: \sigma^2 \neq \sigma_0^2$



Reject H_0 if $\chi^2_{n-1} > \chi^2_{n-1,\frac{\alpha}{2}}$ or $\chi^2_{n-1} < \chi^2_{n-1,1-\frac{\alpha}{2}}$

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Chapter Summary

- Addressed hypothesis testing methodology
- Performed *z* Test for the mean (σ known)
- Discussed critical value and *p*-value approaches to hypothesis testing
- Performed one-tail and two-tail tests
- Performed *t* test for the mean (σ unknown)
- Performed *z* test for the proportion
- Discussed type II error and power of the test
- Performed a hypothesis test for the variance (χ^2)

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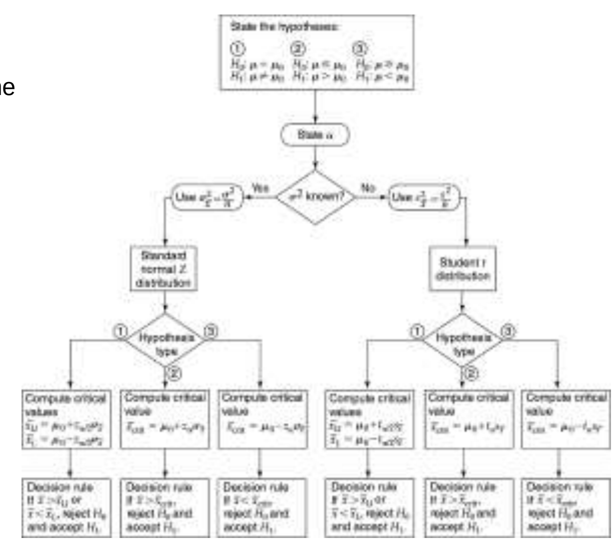
Appendix: Guidelines for Decision Rule (1 of 2)

Figure 9.11

Guidelines for Choosing the Appropriate Decision Rule for a Population Mean



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Appendix: Guidelines for Decision Rule (2 of 2)

Figure 9.12

Guidelines for Choosing the Appropriate Decision Rule for a Population Proportion