Chapter 3

Principles of Option Pricing

Chapter 3: Principles of Option Pricing

Well, it helps to look at derivatives like atoms. Split them one way and you have heat and energy - useful stuff. Split them another way and you have a bomb. You have to understand the subtleties.

Kate Jennings

Moral Hazard, Fourth Estate, 2002, p. 8

Important Concepts in Chapter 3

- Role of arbitrage in pricing options
- Minimum value, maximum value, value at expiration and lower bound of an option price
- Effect of exercise price, time to expiration, risk-free rate and volatility on an option price
- Difference between prices of European and American options
- Put-call parity

Basic Notation and Terminology (1 of 2)

- Symbols
  - S_0 (stock price)
  - X (exercise price)
  - T (time to expiration = (days until expiration)/365)
  - r (see below)
  - S_T (stock price at expiration)
  - $C(S_0,T,X)$, $P(S_0,T,X)$

Basic Notation and Terminology (2 of 2)

- Computation of risk-free rate (r)
  - Date: May 14. Option expiration: May 21
  - T-bill bid discount = 4.45, ask discount = 4.37
    - Average T-bill discount = $(4.45+4.37)/2 = 4.41$
  - T-bill price = $100 - 4.41(7/360) = 99.91425$
  - T-bill yield = $(100/99.91425)^{(365/7)} - 1 = 0.0457$
  - So 4.57% is risk-free rate for options expiring May 21
  - Other risk-free rates: 4.56 (June 18), 4.63 (July 16)
- See Table 3.1 on 24th slide, for prices of DCRB options

Principles of Call Option Pricing (1 of 8)

- Minimum Value of a Call
  - $C(S_0,T,X) \geq 0$ (for any call)
  - For American calls:
    - $C_a(S_0,T,X) \geq \text{Max}(0, S_0 - X)$
  - Concept of intrinsic value: $\text{Max}(0, S_0 - X)$
    - Proof of intrinsic value rule for DCRB calls
  - Concept of time value
    - See Table 3.2 on 25th slide, for time values of DCRB calls
  - See Figure 3.1 on 26th slide, for minimum values of calls

Principles of Call Option Pricing (2 of 8)

- Maximum Value of a Call
  - $C(S_0,T,X) \leq S_0$
  - Intuition
  - See Figure 3.2 on 27th slide, which adds this to Figure 3.1
- Value of a Call at Expiration
  - $C(S_T,0,X) = \text{Max}(0, S_T - X)$
  - Proof/intuition
  - For American and European options
  - See Figure 3.3 on 28th slide

Principles of Call Option Pricing (3 of 8)

- Effect of Time to Expiration
  - Two American calls differing only by time to expiration, T_1 and T_2 where $T_1 < T_2$.
  - $C_a(S_0,T_2,X) \geq C_a(S_0,T_1,X)$
    - Proof/intuition
  - Deep in- and out-of-the-money
  - Time value maximized when at-the-money
  - Concept of time value decay
  - See Figure 3.4 on 29th slide and Table 3.2 on 25th slide
  - Cannot be proven (yet) for European calls

Principles of Call Option Pricing (4 of 8)

- Effect of Exercise Price
  - Effect on Option Value
    - Two European calls differing only by strikes of X_1 and X_2. Which is greater, $C_e(S_0,T,X_1)$ or $C_e(S_0,T,X_2)$?
    - Construct portfolios A and B. See Table 3.3 on 30th slide.
    - Portfolio A has non-negative payoff; therefore,
      - $C_e(S_0,T,X_1) \geq C_e(S_0,T,X_2)$
      - Intuition: show what happens if not true
    - Prices of DCRB options conform

Principles of Call Option Pricing (5 of 8)

- Effect of Exercise Price (continued)
  - Limits on the Difference in Premiums
    - Again, note Table 3.3 on 30th slide. We must have
      - $(X_2 - X_1)(1+r)^{-T} \geq C_e(S_0,T,X_1) - C_e(S_0,T,X_2)$
      - $X_2 - X_1 \geq C_e(S_0,T,X_1) - C_e(S_0,T,X_2)$
      - $X_2 - X_1 \geq C_a(S_0,T,X_1) - C_a(S_0,T,X_2)$
      - Implications
    - See Table 3.4 on 32nd slide. Prices of DCRB options conform

Principles of Call Option Pricing (6 of 8)

- Lower Bound of a European Call
  - Construct portfolios A and B. See Table 3.5 on 32nd slide.
  - B dominates A. This implies that (after rearranging)
    - $C_e(S_0,T,X) \geq \text{Max}[0,\ S_0 - X(1+r)^{-T}]$
    - This is the lower bound for a European call
    - See Figure 3.5 on 33rd slide, for the price curve for European calls
  - Dividend adjustment: subtract present value of dividends from S_0; adjusted stock price is S_0'
  - For foreign currency calls,
    - $C_e(S_0,T,X) \geq \text{Max}[0,\ S_0(1+\rho)^{-T} - X(1+r)^{-T}]$

Principles of Call Option Pricing (7 of 8)

- American Call Versus European Call
  - $C_a(S_0,T,X) \geq C_e(S_0,T,X)$
  - But $S_0 - X(1+r)^{-T} > S_0 - X$ prior to expiration so
    - $C_a(S_0,T,X) \geq \text{Max}\left(0,\ S_0 - X(1+r)^{-T}\right)$
    - Look at Table 3.6 on 34th slide, for lower bounds of DCRB calls
  - If there are no dividends on the stock, an American call will never be exercised early. It will always be better to sell the call in the market.
    - Intuition

Principles of Call Option Pricing (8 of 8)

- Early Exercise of American Calls on Dividend-Paying Stocks
  - If a stock pays a dividend, it is possible that an American call will be exercised as close as possible to the ex-dividend date. (For a currency, the foreign interest can induce early exercise.)
  - Intuition
- Effect of Interest Rates
- Effect of Stock Volatility

Principles of Put Option Pricing (1 of 10)

- Minimum Value of a Put
  - $P(S_0,T,X) \geq 0$ (for any put)
  - For American puts:
    - $P_a(S_0,T,X) \geq \text{Max}(0, X - S_0)$
  - Concept of intrinsic value: $\text{Max}(0, X - S_0)$
    - Proof of intrinsic value rule for DCRB puts
  - See Figure 3.6 on 35th slide, for minimum values of puts
  - Concept of time value
    - See Table 3.7 on 36th slide, on for time values of DCRB puts

Principles of Put Option Pricing (2 of 10)

- Maximum Value of a Put
  - $P_e(S_0, T, X) \leq X(1+r)^{-T}$
  - $P_a(S_0,T,X) \leq X$
  - Intuition
  - See Figure 3.7 on 37th slide, which adds this to Figure 3.6 on 35th slide
- Value of a Put at Expiration
  - $P(S_T,0,X) = \text{Max}(0, X - S_T)$
  - Proof/intuition
  - For American and European options
  - See Figure 3.8 on 38th slide

Principles of Put Option Pricing (3 of 10)

- Effect of Time to Expiration
  - Two American puts differing only by time to expiration, T_1 and T_2 where $T_1 < T_2$.
  - $P_a(S_0,T_2,X) \geq P_a(S_0,T_1,X)$
    - Proof/intuition
  - See Figure 3.9 on 39th slide and Table 3.7 on 36th slide
  - Cannot be proven for European puts

Principles of Put Option Pricing (4 of 10)

- Effect of Exercise Price
  - Effect on Option Value
    - Two European puts differing only by X_1 and X_2. Which is greater, $P_e(S_0,T,X_1)$ or $P_e(S_0,T,X_2)$?
    - Construct portfolios A and B. See Table 3.8 on 40th slide.
    - Portfolio A has non-negative payoff; therefore,
      - $P_e(S_0,T,X_2) \geq P_e(S_0,T,X_1)$
      - Intuition: show what happens if not true
    - Prices of DCRB options conform

Principles of Put Option Pricing (5 of 10)

- Effect of Exercise Price (continued)
  - Limits on the Difference in Premiums
    - Again, note Table 3.8 on 40th slide. We must have
      - $(X_2 - X_1)(1+r)^{-T} \geq P_e(S_0,T,X_2) - P_e(S_0,T,X_1)$
      - $X_2 - X_1 \geq P_e(S_0,T,X_2) - P_e(S_0,T,X_1)$
      - $X_2 - X_1 \geq P_a(S_0,T,X_2) - P_a(S_0,T,X_1)$
      - Implications
    - See Table 3.9 on 41st slide. Prices of DCRB options conform

Principles of Put Option Pricing (6 of 10)

- Lower Bound of a European Put
  - Construct portfolios A and B. See Table 3.10 on 42nd slide.
  - A dominates B. This implies that (after rearranging)
    - $P_e(S_0,T,X) \geq \text{Max}(0,\ X(1+r)^{-T} - S_0)$
    - This is the lower bound for a European put
    - See Figure 3.10 on 43rd slide, for the price curve for European puts
  - Dividend adjustment: subtract present value of dividends from S to obtain S′

Principles of Put Option Pricing (7 of 10)

- American Put Versus European Put
  - $P_a(S_0,T,X) \geq P_e(S_0,T,X)$
- Early Exercise of American Puts
  - There is always a sufficiently low stock price that will make it optimal to exercise an American put early.
  - Dividends on the stock reduce the likelihood of early exercise.

Principles of Put Option Pricing (8 of 10)

- Put-Call Parity
  - Form portfolios A and B where the options are European. See Table 3.11 on 44th slide.
  - The portfolios have the same outcomes at the options' expiration. Thus, it must be true that
    - $S_0 + P_e(S_0, T, X) = C_e(S_0, T, X) + X(1+r)^{-T}$
    - This is called put-call parity.
    - It is important to see the alternative ways the equation can be arranged and their interpretations.

Principles of Put Option Pricing (9 of 10)

- Put-call parity for American options can be stated only as inequalities:

$$C_a(S_0', T, X) + X + \sum_{j=1}^{N} D_j(1+r)^{-t_j}$$
$$\geq S_0 + P_a(S_0', T, X)$$
$$\geq C_a(S_0', T, X) + X(1+r)^{-T}$$

- See Table 3.12 on 45th slide, for put-call parity for DCRB options
- See Figure 3.11 on 47th slide, for linkages between underlying asset, risk-free bond, call, and put through put-call parity.

Principles of Put Option Pricing (10 of 10)

- The Effect of Interest Rates
- The Effect of Stock Volatility

Summary
See Table 3.13 on 48th slide.

Appendix 3: The Dynamics of Option Boundary Conditions: A Learning Exercise

TABLE 3.1: DCRB OPTION DATA, MAY 14

EXERCISE PRICE	CALLS IN MAY	CALLS IN JUNE	CALLS IN JULY	PUTS IN MAY	PUTS IN JUNE	PUTS IN JULY
120	8.75	15.40	20.90	2.75	9.25	13.65
125	5.75	13.50	18.60	4.60	11.50	16.60
130	3.60	11.35	16.40	7.35	14.25	19.65

Current stock price: 125.94

Expirations: May 21, June 18, July 16

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TABLE 3.2: INTRINSIC VALUES AND TIME VALUES OF DCRB CALLS

EXERCISE PRICE	INTRINSIC VALUE	TIME VALUE OF MAY	TIME VALUE OF JUNE	TIME VALUE OF JULY
120	5.94	2.81	9.46	14.96
125	0.94	4.81	12.56	17.66
130	0.00	3.60	11.35	16.40

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FIGURE 3.1: Minimum Values of European and American Calls

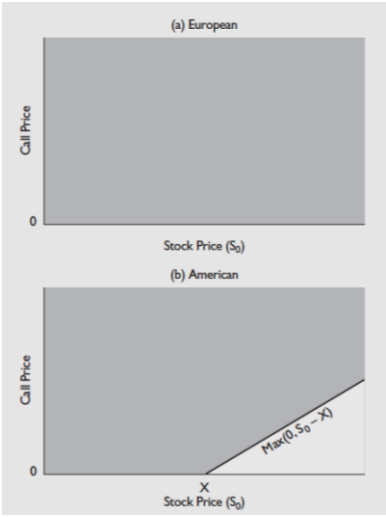

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FIGURE 3.2: Minimum And Maximum Values of European and American Calls

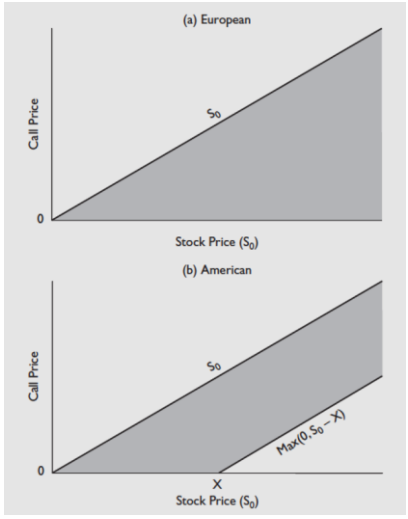

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FIGURE 3.3: The Value of a Call at Expiration

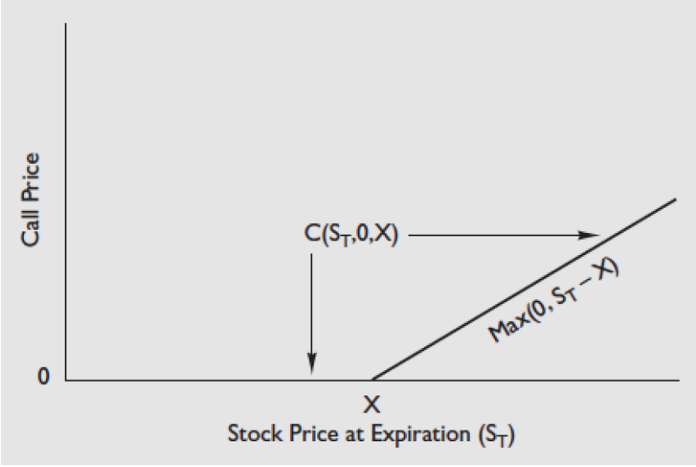

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FIGURE 3.4: The Price Curve for American Calls

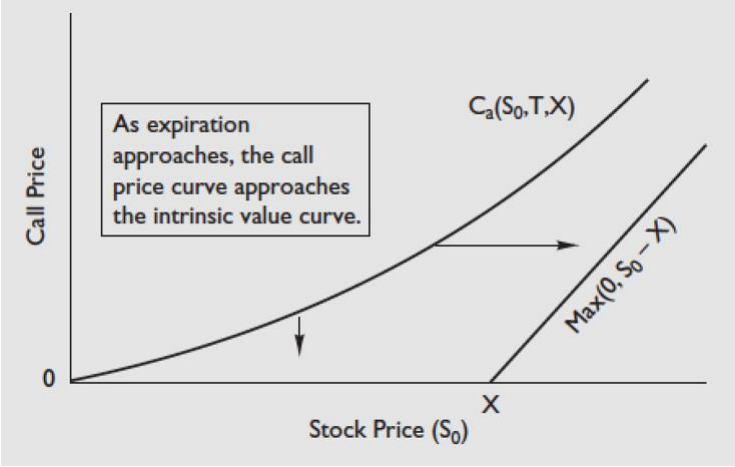

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TABLE 3.3: THE EFFECT OF EXERCISE PRICE ON CALL VALUE: PAYOFFS AT EXPIRATION OF PORTFOLIOS A AND B

		PAYOFFS FROM PORTFOLIO GIVEN STOCK PRICE AT EXPIRATION		
PORTFOLIO	CURRENT VALUE	$S_T < X_1$	$X_1 \leq S_T < X_2$	$X_1 < X_2 \leq S_T$
A	$+C_e(S_0,T,X_1)$	0	$S_T - X_1$	$S_T - X_1$
	$-C_e(S_0,T,X_2)$	0	0	$-S_T + X_2$
		0	$S_T - X_1 \geq 0$	$X_2 - X_1 > 0$
			$X_2 - X_1 > 0$	$X_2 - X_1 > 0$
B	$(X_2 - X_1)(1 + r)^{-T}$	$X_2 - X_1 > 0$	$X_2 - X_1 > 0$	$X_2 - X_1 > 0$

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TABLE 3.4: THE RELATIONSHIP BETWEEN EXERCISE PRICE AND CALL PRICE FOR DCRB CALLS

EXERCISE PRICES	EXERCISE PRICE DIFFERENCE	DIFFERENCE BETWEEN CALL PRICES (PRESENT VALUE OF DIFFERENCE BETWEEN EXERCISE PRICES IN PARENTHESES) IN MAY	DIFFERENCE BETWEEN CALL PRICES (PRESENT VALUE OF DIFFERENCE BETWEEN EXERCISE PRICES IN PARENTHESES) IN JUNE	DIFFERENCE BETWEEN CALL PRICES (PRESENT VALUE OF DIFFERENCE BETWEEN EXERCISE PRICES IN PARENTHESES) IN JULY
120, 125	5	3.00 (4.9957)	1.90 (4.9787)	2.30 (4.9611)
120, 130	10	5.15 (9.9914)	4.05 (9.9573)	4.50 (9.9222)
125, 130	5	2.15 (4.9957)	2.15 (4.9787)	2.20 (4.9611)

Note: Risk-free rates are 4.57% (May), 4.56% (June), and 4.63% (July); times to expiration are 0.0192 (May), 0.0959 (June), and 0.1726 (July).

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TABLE 3.5: THE LOWER BOUND OF A EUROPEAN CALL: PAYOFFS AT EXPIRATION OF PORTFOLIOS A AND B

PORTFOLIO	CURRENT VALUE	PAYOFFS FROM PORTFOLIO GIVEN STOCK PRICE AT EXPIRATION	
		$S_T \leq X$	$S_T > X$
A	S_0	S_T	S_T
B	$C_e(S_0,T,X) + X(1 + r)^{-T}$	X	$(S_T - X) + X = S_T$

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FIGURE 3.5: The Price Curve for European Calls

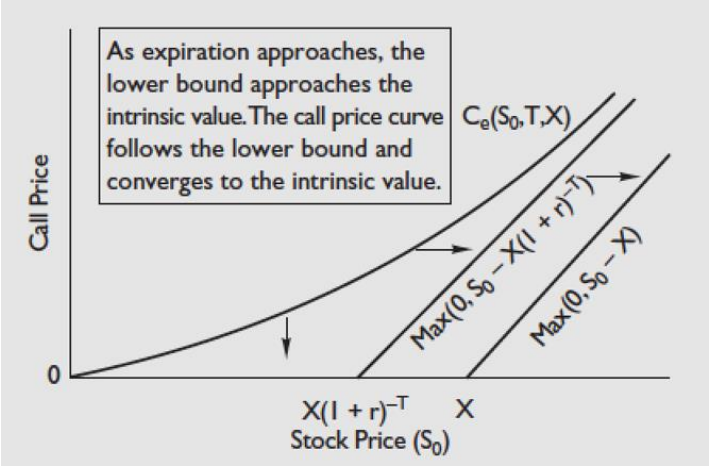

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TABLE 3.6: LOWER BOUNDS OF DCRB CALLS

EXERCISE PRICE	EXPIRATION IN MAY	EXPIRATION IN JUNE	EXPIRATION IN JULY
120	6.0428	6.4520	6.8738
125	1.0471	1.4733	1.9127
130	0.0000	0.0000	0.0000

Note: Risk-free rates are 4.57% (May), 4.56% (June), and 4.63% (July); times to expiration are 0.0192 (May), 0.0959 (June), and 0.1726 (July).

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FIGURE 3.6: Minimum Value of European and American Puts

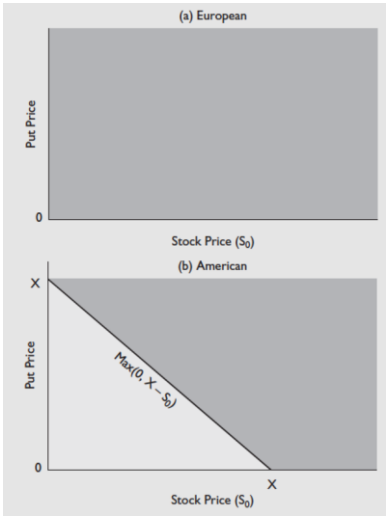

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TABLE 3.7: INTRINSIC VALUES AND TIME VALUES OF DCRB PUTS

EXERCISE PRICE	INTRINSIC VALUE	TIME VALUE IN MAY	TIME VALUE IN JUNE	TIME VALUE IN JULY
120	0.00	2.75	9.25	13.65
125	0.00	4.60	11.50	16.60
130	4.06	3.29	10.19	15.59

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FIGURE 3.7: Minimum and Maximum Values of European and American Puts

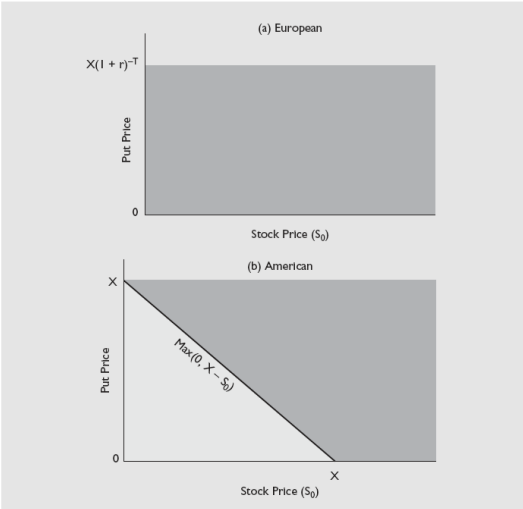

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FIGURE 3.8: The Value of a Put at Expiration

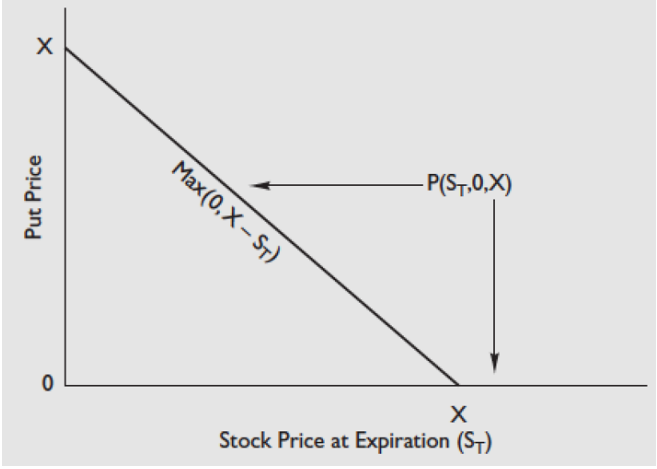

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FIGURE 3.9: The Price Curve for American Puts

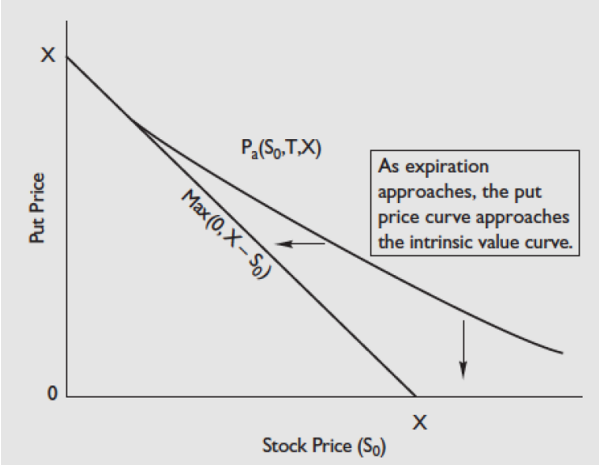

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TABLE 3.8: THE EFFECT OF EXERCISE PRICE ON PUT VALUE: PAYOFFS AT EXPIRATION OF PORTFOLIOS A AND B

		PAYOFFS FROM PORTFOLIO GIVEN STOCK PRICE AT EXPIRATION		
PORTFOLIO	CURRENT VALUE	$S_T < X_1$	$X_1 \leq S_T < X_2$	$X_1 < X_2 \leq S_T$
A	$-P_e(S_0,T,X_1)$	$-X_1 + S_T$	0	0
	$+P_e(S_0,T,X_2)$	$X_2 - S_T$	$X_2 - S_T$	0
		$X_2 - X_1 > 0$	$X_2 - S_T > 0$	0
B	$(X_2 - X_1)(1 + r)^{-T}$	$X_2 - X_1 > 0$	$X_2 - X_1 > 0$	$X_2 - X_1 > 0$

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TABLE 3.9: THE RELATIONSHIP BETWEEN EXERCISE PRICE AND PUT PRICE FOR DCRB PUTS

EXERCISE PRICES	EXERCISE PRICE DIFFERENCE	DIFFERENCE BETWEEN PUT PRICES (PRESENT VALUE OF DIFFERENCE BETWEEN EXERCISE PRICES IN PARENTHESES) ON MAY	DIFFERENCE BETWEEN PUT PRICES (PRESENT VALUE OF DIFFERENCE BETWEEN EXERCISE PRICES IN PARENTHESES) ON JUNE	DIFFERENCE BETWEEN PUT PRICES (PRESENT VALUE OF DIFFERENCE BETWEEN EXERCISE PRICES IN PARENTHESES) ON JULY
120, 125	5	1.85 (4.9957)	2.25 (4.9787)	2.95 (4.9611)
120, 130	10	4.60 (9.9914)	5.00 (9.9573)	6.00 (9.9222)
125, 130	5	2.75 (4.99571)	2.75 (4.97871)	3.05 (4.96111)

Note: Risk-free rates are 4.57% (May), 4.56% (June), and 4.63% (July); times to expiration are 0.0192 (May), 0.0959 (June), and 0.1726 (July).

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TABLE 3:10 LOWER BOUND OF A EUROPEAN PUT: PAYOFFS AT EXPIRATION OF PORTFOLIOS A AND B

PORTFOLIO	CURRENT VALUE	PAYOFFS FROM PORTFOLIO GIVEN STOCK PRICE AT EXPIRATION	
		$S_T < X$	$S_T \geq X$
A	S_0	S_T	S_T
B	$X(1 + r)^{-T} - P_e(S_0,T,X)$	$X - (X - S_T) = S_T$	X

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FIGURE 3.10: The Price Curve for European Puts

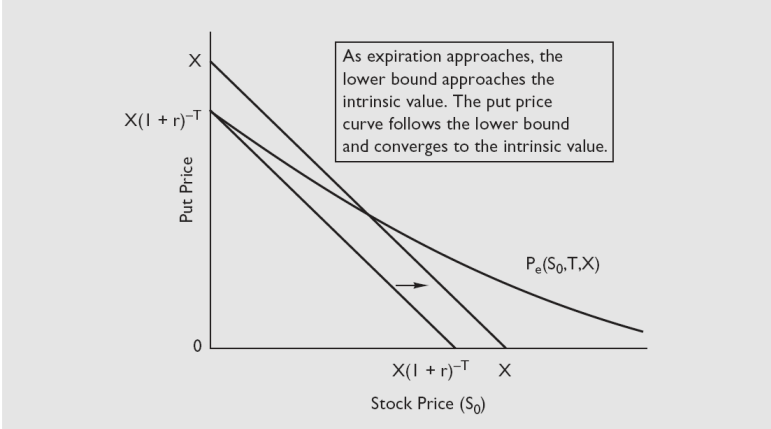

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TABLE 3.11: PUT–CALL PARITY

		PAYOFFS FROM PORTFOLIO GIVEN STOCK PRICE AT EXPIRATION	
PAYOFF FROM	CURRENT VALUE	$S_T \leq X$	$S_T > X$
A Stock	S_0	S_T	S_T
Put	$P_e(S_0,T,X)$	$X - S_T$	0
		X	S_T
B Call	$C_e(S_0,T,X)$	0	$S_T - X$
Bonds	$X(1 + r)^{-T}$	X	X
		X	S_T

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TABLE 3.12: PUT−CALL PARITY FOR DCRB CALLS (1 of 2)

A European Put−Call Parity

Top row of cell: $S_0 + P_e(S_0,T,X)$

Bottom row of cell: $C_e(S_0,T,X) + X(1+r)^{-T}$

EXERCISE PRICE	MAY	JUNE	JULY
120	128.69 128.6472	135.19 134.8880	139.59 139.9662
125	130.54 130.6429	137.44 137.9667	142.54 142.6273
130	133.29 133.4886	140.19 140.7953	145.59 145.3884

(To continued)

TABLE 3.12: PUT−CALL PARITY FOR DCRB CALLS (2 of 2)

B American Put−Call Parity

Top row of cell: $C_a(S_0,T,X) + X$ (Note: No dividends, so $S_0' = S_0$.)

Middle row of cell: $S_0 + P_a(S_0,T,X)$

Bottom row of cell: $C_a(S_0,T,X) + X(1+r)^{-T}$

EXERCISE PRICE	MAY	JUNE	JULY
120	128.75 128.69 128.6472	135.40 135.19 134.8880	140.90 139.59 139.9662
125	130.75 130.54 130.6429	138.50 137.44 137.9667	143.60 142.54 142.6273
130	133.60 133.29 133.4886	141.35 140.19 140.7953	146.40 145.59 145.3884

Note: Risk-free rates are 4.57% (May), 4.56% (June), and 4.63% (July); times to expiration are 0.0192 (May), 0.0959 (June), and 0.1726 (July).

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FIGURE 3.11: The Linkage between Calls, Puts, Underlying Asset, and Risk-Free Bonds

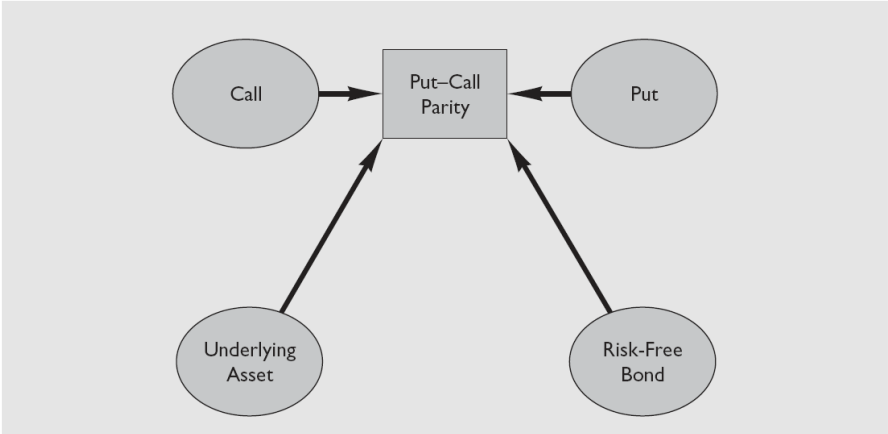

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TABLE 3.13: SUMMARY OF THE PRINCIPLES OF OPTION PRICING

	EUROPEAN CALLS	AMERICAN CALLS	EUROPEAN PUTS	AMERICAN PUTS
Assuming no dividends				
Minimum value	≥ 0	≥ 0	≥ 0	≥ 0
Intrinsic value (IV)	NA	$\text{Max}(0, S_0 - X)$	NA	$\text{Max}(0, X - S_0)$
Time value	$C_e(S_0,T,X) - \text{Max}(0, S_0 - X)$	$C_a(S_0,T,X) - IV$	$P_e(S_0,T,X) - \text{Max}(0, X - S_0)$	$P_a(S_0,T,X) - IV$
Maximum value	S_0	S_0	$X(1 + r)^{-T}$	X
Effect of time to expiration	$C_e(S_0,T_2,X) \geq C_e(S_0,T_1,X)$	$C_a(S_0,T_2,X) \geq C_a(S_0,T_1,X)$	$P_e(S_0,T_2,X_2) \gtrless P_e(S_0,T_1,X)$	$P_a(S_0,T_2,X) \geq P_a(S_0,T_1,X)$
Effect of exercise price	$C_e(S_0,T,X_1) \geq C_e(S_0,T,X_2)$	$C_a(S_0,T,X_1) \geq C_a(S_0,T,X_2)$	$P_e(S_0,T,X_2) \geq P_e(S_0,T,X_1)$	$P_a(S_0,T,X_2) \geq P_a(S_0,T,X_1)$
Maximum difference in premiums	$(X_2 - X_1)(1 + r)^{-T}$	$X_2 - X_1$	$(X_2 - X_1)(1 + r)^{-T}$	$X_2 - X_1$
Lower bound	$\text{Max}[0, S_0 - X(1 + r)^{-T}]$	$\text{Max}[0, S_0 - X(1 + r)^{-T}]$	$\text{Max}[0, X(1 + r)^{-T} - S_0]$	$\text{Max}(0, X - S_0)$
Other results				
American versus	$C_a(S_0,T,X) \geq C_e(S_0,T,X)$			
European option prices	$P_a(S_0,T,X) \geq P_e(S_0,T,X)$			
Put–call parity	$P_e(S_0,T,X) = C_e(S_0,T,X) - S_0 + X(1 + r)^{-T}$ (European) $C_a(S_0,T,X) + X \geq S_0 + P_a(S_0,T,X) \geq C_a(S_0,T,X) + X(1 + r)^{-T}$ (American)			
Effect of dividends	Replace S_0 with $S'_0 = S_0 - \sum_{i=1}^{N} D_i(1 + r)^{-t_i}$, and for American put–call parity, add $\sum_{i=1}^{N} D_i(1 + r)^{-t_i}$ to the left-hand term. The middle S_0 does not have the dividend adjustment.			
Currency options	Replace S_0 with $S_0(1 + \rho)^{-T}$.			

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