Chapter 4

Option Pricing Models: The Binomial Model

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Chapter 4: Option Pricing Models: The Binomial Model

Options traders can get by with less math than you think. Tour de France cyclists don't need to know how to solve Newton's laws in order to bank around a curve. Indeed, thinking too much about physics while riding or playing tennis may prove a hindrance. But good traders do have to have the patience to understand the essential mechanism of replicating the factors they're trading.

Emanuel Derman

The Journal of Derivatives, Winter 2000, p. 62

Important Concepts in Chapter 4 (1 of 2)

- The concept of an option pricing model
- The one- and two-period binomial option pricing models
- Explanation of the establishment and maintenance of a risk-free hedge
- Illustration of how early exercise can be captured
- The extension of the binomial model to any number of time periods
- Alternative specifications of the binomial model

Important Concepts in Chapter 4 (2 of 2)

- Definition of a model
  - A simplified representation of reality that uses certain inputs to produce an output or result
- Definition of an option pricing model
  - A mathematical formula that uses the factors that determine an option's price as inputs to produce the theoretical fair value of an option.

One-Period Binomial Model (1 of 8)

- Conditions and assumptions
  - One period, two outcomes (states)
  - S = current stock price
  - u = 1 + return if stock goes up
  - d = 1 + return if stock goes down
  - r = risk-free rate
- Value of European call at expiration one period later
  - $C_u = \text{Max}(0, Su - X)$ or
  - $C_d = \text{Max}(0, Sd - X)$
- See Figure 4.1 on 49th slide.

One-Period Binomial Model (2 of 8)

- Important point: $d < 1 + r < u$ to prevent arbitrage
- We construct a hedge portfolio of h shares of stock and one short call. Current value of portfolio:
  - $V = hS - C$
- At expiration the hedge portfolio will be worth
  - $V_u = hSu - C_u$
  - $V_d = hSd - C_d$
  - If we are hedged, these must be equal. Setting $V_u = V_d$ and solving for h gives

One-Period Binomial Model (3 of 8)

$$h = \frac{C_u - C_d}{Su - Sd}$$

- These values are all known so h is easily computed
- Since the portfolio is riskless, it should earn the risk-free rate. Thus
  - $V(1 + r) = V_u$ (or V_d)
- Substituting for V and V_u
  - $(hS - C)(1 + r) = hSu - C_u$
- And the theoretical value of the option is

One-Period Binomial Model (4 of 8)

$$C = \frac{pC_u + (1-p)C_d}{1+r}$$

where

$$p = (1+r-d)/(u-d)$$

- This is the theoretical value of the call as determined by the stock price, exercise price, risk-free rate, and up and down factors.
- The probabilities of the up and down moves were never specified. They are irrelevant to the option price.

One-Period Binomial Model (5 of 8)

- An Illustrative Example
  - $S = 100$, $X = 100$, $u = 1.25$, $d = 0.80$, $r = 0.07$
  - First find the values of C_u, C_d, h, and p:
    - $C_u = \text{Max}(0, 100(1.25) - 100)$
      $= \text{Max}(0, 125 - 100) = 25$
    - $C_d = \text{Max}(0, 100(0.80) - 100)$
    - $= \text{Max}(0, 80 - 100) = 0$
    - $h = (25 - 0)/(125 - 80) = 0.556$
    - $p = (1.07 - 0.80)/(1.25 - 0.80) = 0.6$
- Then insert into the formula for C:

$$C = \frac{(0.6)25 + (0.4)0.0}{1.07} = 14.02$$

One-Period Binomial Model (6 of 8)

- A Hedged Portfolio
  - Short 1,000 calls and long 1000h = 1000(0.556) = 556 shares. See Figure 4.2 on 50th slide.
  - Value of investment: V = 556($100) − 1,000($14.02) $41,580. (This is how much money you must put up.)
  - Stock goes to $125
    - Value of investment = 556($125) − 1,000($25) = $44,500
  - Stock goes to $80
    - Value of investment = 556($80) − 1,000($0) = $44,480

One-Period Binomial Model (7 of 8)

You invested $41,580 and got back $44,500, a 7 % return, which is the risk-free rate.

- An Overpriced Call
  - Let the call be selling for $15.00
  - Your amount invested is 556($100) − 1,000($15.00) = $40,600
  - You will still end up with $44,500, which is a 9.6% return.
  - Everyone will take advantage of this, forcing the call price to fall to $14.02

One-Period Binomial Model (8 of 8)

- An Underpriced Call
  - Let the call be priced at $13
  - Sell short 556 shares at $100 and buy 1,000 calls at $13. This will generate a cash inflow of $42,600.
  - At expiration, you will end up paying out $44,500.
  - This is like a loan in which you borrowed $42,600 and paid back $44,500, a rate of 4.46%, which beats the risk-free borrowing rate.

Two-Period Binomial Model (1 of 10)

- We now let the stock go up another period so that it ends up Su^2, Sud or Sd^2.
- See Figure 4.3 on 51th slide.
- The option expires after two periods with three possible values:

$$C_{u^2} = \text{Max}[0, Su^2 - X]$$
$$C_{ud} = \text{Max}[0, Sud - X]$$
$$C_{d^2} = \text{Max}[0, Sd^2 - X]$$

Two-Period Binomial Model (2 of 10)

- After one period the call will have one period to go before expiration. Thus, it will worth either of the following two values

$$C_u = \frac{pC_{u^2} + (1-p)C_{ud}}{1+r}, \text{or}$$

$$C_d = \frac{pC_{du} + (1-p)C_{d^2}}{1+r}$$

- The price of the call today will be

Two-Period Binomial Model (3 of 10)

$$C = \frac{pC_u + (1-p)C_d}{1+r}$$

which can also be written as

$$C = \frac{p^2C_{u^2} + 2p(1-p)C_{ud} + (1-p)^2C_{d^2}}{(1+r)^2}$$

- The hedge ratios are different in the different states:

$$h = \frac{C_u - C_d}{Su - Sd}, \; h_u = \frac{C_{u^2} - C_{ud}}{Su^2 - Sud}, \; h_d = \frac{C_{ud} - C_{d^2}}{Sud - Sd^2}$$

Two-Period Binomial Model (4 of 10)

- An Illustrative Example
  - $Su^2 = 100(1.25)^2 = 156.25$
  - Sud = 100(1.25)(0.80) = 100
  - $Sd^2 = 100(0.80)^2 = 64$
  - The call option prices are as follows

$$C_{u^2} = \text{Max}[0, Su^2 - X] = \text{Max}[0, 156.25 - 100] = 56.25$$

$$C_{ud} = \text{Max}[0, Sud - X] = \text{Max}[0, 100 - 100] = 0.0$$

$$C_{d^2} = \text{Max}[0, Sd^2 - X] = \text{Max}[0, 64 - 100] = 0.0$$

Two-Period Binomial Model (5 of 10)

- The two values of the call at the end of the first period are

$$C_u = \frac{pC_{u^2} + (1-p)C_{ud}}{1+r} = \frac{(0.6)56.25+(0.4)0.0}{1.07} = 31.54$$

$$\text{or } C_d = \frac{pC_{du} + (1-p)C_{d^2}}{1+r} = \frac{(0.6)0.0+(0.4)0.0}{1.07} = 0.0$$

Two-Period Binomial Model (6 of 10)

- Therefore, the value of the call today is

$$C = \frac{pC_u + (1-p)C_d}{1+r}$$

$$= \frac{(0.6)31.54 + (0.4)0.0}{1.07} = 17.69$$

Two-Period Binomial Model (7 of 10)

- A Hedge Portfolio
  - See Figure 4.4 on 52nd slide.
  - Call trades at its theoretical value of $17.69.
  - Hedge ratio today: $h = (31.54 - 0.0)/(125 - 80) = 0.701$
  - So
    - Buy 701 shares at $100 for $70,100
    - Sell 1,000 calls at $17.69 for $17,690
    - Net investment: $52,410

Two-Period Binomial Model (8 of 10)

- A Hedge Portfolio (continued)
  - Note each of the possibilities:
    - Stock goes to 125, then 156.25
    - Stock goes to 125, then to 100
    - Stock goes to 80, then to 100
    - Stock goes to 80, then to 64
  - In each case, you wealth grows by 7% at the end of the first period. You then revise the mix of stock and calls by either buying or selling shares or options. Funds realized from selling are invested at 7% and funds necessary for buying are borrowed at 7%.

Two-Period Binomial Model (9 of 10)

- A Hedge Portfolio (continued)
  - Your wealth then grows by 7% from the end of the first period to the end of the second.
  - Conclusion: If the option is correctly priced and you maintain the appropriate mix of shares and calls as indicated by the hedge ratio, you earn a risk-free return over both periods.

Two-Period Binomial Model (10 of 10)

- A Mispriced Call in the Two-Period World
  - If the call is underpriced, you buy it and short the stock, maintaining the correct hedge over both periods. You end up borrowing at less than the risk-free rate.
  - If the call is overpriced, you sell it and buy the stock, maintaining the correct hedge over both periods. You end up lending at more than the risk-free rate.
  - See Table 4.1 on 53rd slide for summary.

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Extensions of the Binomial Model (1 of 25)

- Pricing Put Options
  - Same procedure as calls but use put payoff formula at expiration. In our example the put prices at expiration are

$$P_{u^2} = \text{Max}[0, X - Su^2] = \text{Max}[0,\ 100 - 156.25] = 0.0$$

$$P_{ud} = \text{Max}[0, X - Sud] = \text{Max}[0,\ 100 - 100] = 0.0$$

$$P_{d^2} = \text{Max}[0, X - Sd^2] = \text{Max}[0,\ 100 - 64] = 36.0$$

Extensions of the Binomial Model (2 of 25)

- Pricing Put Options (continued)
  - The two values of the put at the end of the first period are

$$P_u = \frac{pP_{u^2} + (1-p)P_{ud}}{1+r} = \frac{(0.6)0.0+(0.4)0.0}{1.07} = 0.0,$$

$$\text{or } P_d = \frac{pP_{du} + (1-p)P_{d^2}}{1+r} = \frac{(0.6)0.0 + (0.4)36}{1.07} = 13.46$$

Extensions of the Binomial Model (3 of 25)

- Pricing Put Options (continued)
  - Therefore, the value of the put today is

$$P = \frac{pP_u + (1-p)P_d}{1+r}$$

$$= \frac{(0.6)0.0 + (0.4)13.46}{1.07} = 5.03$$

Extensions of the Binomial Model (4 of 25)

- Pricing Put Options (continued)
  - Let us hedge a long position in stock by purchasing puts. The hedge ratio formula is the same except that we ignore the negative sign:

$$h = \frac{0 - 13.46}{125 - 80} = -0.299$$

  - Thus, we shall buy 299 shares and 1,000 puts. This will cost \$29,900 (299 × \$100) + \$5,030 (1,000 × \$5.03) for a total of \$34,930.

Extensions of the Binomial Model (5 of 25)

- Pricing Put Options (continued)
  - Stock goes from 100 to 125. We now have
    - 299 shares at \$125 + 1,000 puts at \$0.0 = \$37,375
    - This is a 7% gain over \$34,930. The new hedge ratio is

$$h = \frac{0.0 - 0.0}{156.25 - 100} = 0.000$$

  - Thus, sell 299 shares, receiving 299(\$125) = \$37,375, which is invested in risk-free bonds.

Extensions of the Binomial Model (6 of 25)

- Pricing Put Options (continued)
  - Stock goes from 100 to 80. We now have
    - 299 shares at \$80 + 1,000 puts at \$13.46 = \$37,380
    - This is a 7% gain over \$34,930. The new hedge ratio is

$$h = \frac{0 - 36}{100 - 64} = -1.000$$

  - Thus, buy 701 shares, paying 701(\$80) = \$56,080, by borrowing at the risk-free rate.

Extensions of the Binomial Model (7 of 25)

- Pricing Put Options (continued)
  - Stock goes from 125 to 156.25. We now have
    - Bond worth \$37,375(1.07) = \$39,991
    - This is a 7% gain.
  - Stock goes from 125 to 100. We now have
    - Bond worth \$37,375(1.07) = \$39,991
    - This is a 7% gain.

Extensions of the Binomial Model (8 of 25)

- Pricing Put Options (continued)
  - Stock goes from 80 to 100. We now have
    - 1,000 shares worth $100 each, 1,000 puts worth $0 each, plus a loan in which we owe $56,080(1.07) = $60,006 for a total of $39,994, a 7% gain
  - Stock goes from 80 to 64. We now have
    - 1,000 shares worth $64 each, 1,000 puts worth $36 each, plus a loan in which we owe $56,080(1.07) = $60,006 for a total of $39,994, a 7% gain

Extensions of the Binomial Model (9 of 25)

- American Puts and Early Exercise
  - Now we must consider the possibility of exercising the put early. At time 1 the European put values were
    - P_u = 0.00 when the stock is at 125
    - P_d = 13.46 when the stock is at 80
  - When the stock is at 80, the put is in-the-money by $20 so exercise it early. Replace P_u = 13.46 with P_u = 20. The value of the put today is higher at

$$P = \frac{(0.6)0.0 + (0.4)20}{1.07} = 7.48$$

Extensions of the Binomial Model (10 of 25)

- Dividends, European Calls, American Calls, and Early Exercise
  - One way to incorporate dividends is to assume a constant yield, δ, per period. The stock moves up, then drops by the rate δ.
  - See Figure 4.5 on 54th slide for example with a 10% yield
  - The call prices at expiration are

$$C_{u^2} = \text{Max}(0,\ 140.625 - 100) = 40.625$$

$$C_{ud} = C_{du} = \text{Max}(0,\ 90 - 100) = 0.0$$

$$C_{d^2} = \text{Max}(0,\ 57.60 - 100) = 0$$

Extensions of the Binomial Model (11 of 25)

- Dividends, European Calls, American Calls, and Early Exercise (continued)
  - The European call prices after one period are

$$C_u = \frac{(0.6)40.625 + (0.4)0.0}{1.07} = 22.78$$

$$C_u = \frac{(0.6)0.0 + (0.4)0.0}{1.07} = 0.00$$

  - The European call value at time 0 is

$$C = \frac{(0.6)22.78 + (0.4)0.0}{1.07} = 12.77$$

Extensions of the Binomial Model (12 of 25)

- Dividends, European Calls, American Calls, and Early Exercise (continued)
  - If the call is American, when the stock is at 125, it pays a dividend of $12.50 and then falls to $112.50. We can exercise it, paying $100, and receive a stock worth $125. The stock goes ex-dividend, falling to $112.50 but we get the $12.50 dividend. So at that point, the option is worth $25. We replace the binomial value of C_u = $22.78 with C_u = $25. At time 0 the value is

$$C = \frac{(0.6)25 + (0.4)0.0}{1.07} = 14.02$$

Extensions of the Binomial Model (13 of 25)

- Dividends, European Calls, American Calls, and Early Exercise (continued)
  - Alternatively, we can specify that the stock pays a specific dollar dividend at time 1. Assume $12. Unfortunately, the tree no longer recombines, as in Figure 4.6 on 55th slide. We can still calculate the option value but the tree grows large very fast. See Figure 4.7 on 56th slide.
  - Because of the reduction in the number of computations, trees that recombine are preferred over trees that do not recombine.

Extensions of the Binomial Model (14 of 25)

- Dividends, European Calls, American Calls, and Early Exercise (continued)
  - Yet another alternative (and preferred) specification is to subtract the present value of the dividends from the stock price (as we did in Chapter 3) and let the adjusted stock price follow the binomial up and down factors. For this problem, see Figure 4.8 on 57th slide.
  - The tree now recombines and we can easily calculate the option values following the same procedure as before.

Extensions of the Binomial Model (15 of 25)

- Dividends, European Calls, American Calls, and Early Exercise (continued)
  - The option prices at expiration are

$$C_{u^2} = \text{Max}(0,\ 138.74 - 100) = 38.74$$
$$C_{ud} = \text{Max}(0,\ 88.79 - 100) = 0.0$$
$$C_{d^2} = \text{Max}(0,\ 56.82 - 100) = 0.0$$

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Extensions of the Binomial Model

- Dividends, European Calls, American Calls, and Early Exercise (continued)
  - At time 1 the option prices are

$$C_u = \frac{(0.6)38.74 + (0.4)0.0}{1.07} = 21.72$$

$$C_d = \frac{(0.6)0.0 + (0.4)0.0}{1.07} = 0.0$$

  - We exercise at time 1 so that C_u is now 22.99. At time 0

$$C = \frac{(0.6)22.99 + (0.4)0.0}{1.07} = 12.89$$

  - The European option value would be 12.18.

Extensions of the Binomial Model (17 of 25)

- Foreign Currency Options
  - Underlying instrument is currency
  - Holding of foreign currency can earn the foreign risk-free interest rate
  - The binomial probability is altered to adjust for the foreign risk-free interest rate effect
  - The binomial probability is

$$p = \frac{\frac{1+r}{1+\rho} - d}{u - d}$$

Extensions of the Binomial Model (18 of 25)

- Extending the Binomial Model to n Periods
  - With n periods to go, the binomial model can be easily extended. There is a long and somewhat complex looking formula in the book. The basic procedure, however, is the same. See Figure 4.9 on 58th slide, in which we see below the stock prices the prices of European and American puts. This illustrates the early exercise possibilities for American puts, which can occur in multiple time periods.
  - At each step, we must check for early exercise by comparing the value if exercised to the value if not exercised and use the higher value of the two.

Extensions of the Binomial Model (19 of 25)

- Behavior of the Binomial Model for Large n and a Fixed Option Life
  - The risk-free rate is adjusted to $(1+r)^{T/n}-1$
  - The up and down parameters are adjusted to

$$u = e^{\sigma\sqrt{T/n}}$$

$$d = 1/u$$

    - where σ is the volatility. Let us price the DCRB June 125 call with one period.

Extensions of the Binomial Model (20 of 25)

- The Behavior of the Binomial Model for Large n and a Fixed Option Life (continued)
  - The parameters are now

$$r = (1.0456)^{0.0959/1} - 1 = .004285$$

$$u = e^{0.83\sqrt{0.0959/1}} = 1.293087$$

$$d = 1/1.293087 = 0.773343$$

  - The new stock prices are
    - Su = 125.9375(1.293087) = 162.8481
    - Sd = 125.9375(0.773343) = 97.3929

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Extensions of the Binomial Model (21 of 25)

- The Behavior of the Binomial Model for Large n and a Fixed Option Life (continued)
  - The new option prices would be
    - C_u = Max(0, 162.8481 − 125) = 37.85
    - C_d = Max(0, 97.3929 − 125) = 0.0
  - p would be (1.004285 − 0.773343)/(1.293087 − 0.773343) = 0.444; 1 − p = 0.556.
  - The price of the option at time 0 is, therefore,

$$C = \frac{(0.444)37.85 + (0.556)0.00}{1.004285} = 16.74$$

Extensions of the Binomial Model (22 of 25)

- The Behavior of the Binomial Model for Large n and a Fixed Option Life (continued)
  - The actual price of the option is 13.50, but obviously one binomial period is not enough.
  - Table 4.2 on 59^{th} slide, shows what happens as we increase the number of binomial periods. The price converges to around 13.56. In Chapter 5, we shall see that this is approximately the Black-Scholes-Merton price.

Extensions of the Binomial Model (23 of 25)

- Alternative Specifications of the Binomial Model
  - We can use a different specification of u, d and p

$$u = e^{(\ln(1+r) - \sigma^2/2)(T/n) + \sigma\sqrt{T/n}}$$

$$d = e^{(\ln(1+r) - \sigma^2/2)(T/n) - \sigma\sqrt{T/n}}$$

$$p = \frac{e^{\sigma^2(T/n)/2} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}$$

  - where ln(1 + r) is the continuously compounded interest rate. Here p will converge to 0.5 as n increases.

Extensions of the Binomial Model (24 of 25)

- Alternative Specifications of the Binomial Model (continued)
  - Now let us price the DCRB June 125 call but use two periods. We have $r = (1.0456)^{0.0959/2} - 1 = 0.0021$. Using our previous formulas,

$$u = e^{0.83\sqrt{0.0959/2}} = 1.1993$$

$$d = 1/1.1993 = 0.8338$$

$$p = \frac{1.0021 - 0.8338}{1.1993 - 0.8338} = .4605$$

Extensions of the Binomial Model (25 of 25)

- Alternative Specifications of the Binomial Model (continued)
  - Now let us use these new formulas:

$$u = e^{(\ln(1.0456)\text{-}0.83^2/2)(0.0959/2)+0.83\sqrt{0.0959/2}} = 1.1822$$

$$d = e^{(\ln(1.0456)\text{-}0.83^2/2)(0.0959/2)-0.83\sqrt{0.0959/2}} = 0.8219$$

$$p = \frac{e^{0.83^2(0.0959/2)/2} - e^{-0.83\sqrt{0.0959/2}}}{e^{0.83\sqrt{0.0959/2}} - e^{-0.83\sqrt{0.0959/2}}} = 0.500251$$

  - We can use 0.5 for p. See Figure 4.10 on 60th slide. The prices are close and will converge when n is large.
  - See BlackScholesMertonBinomial10e.xlsm for software to calculate the binomial model.

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Summary

FIGURE 4.1: The One-Period Binomial Tree

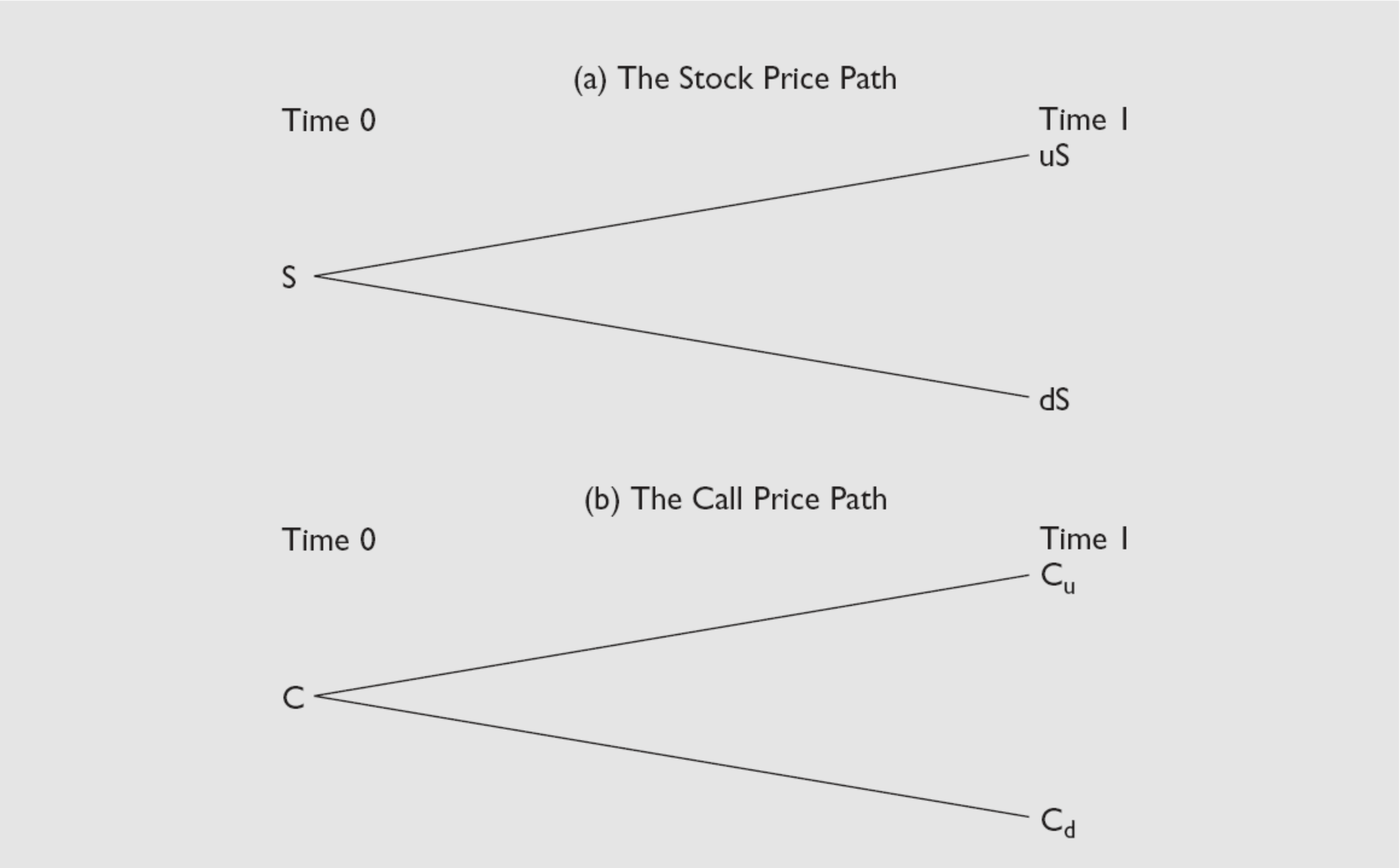

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FIGURE 4.2: The One-Period Binomial Example

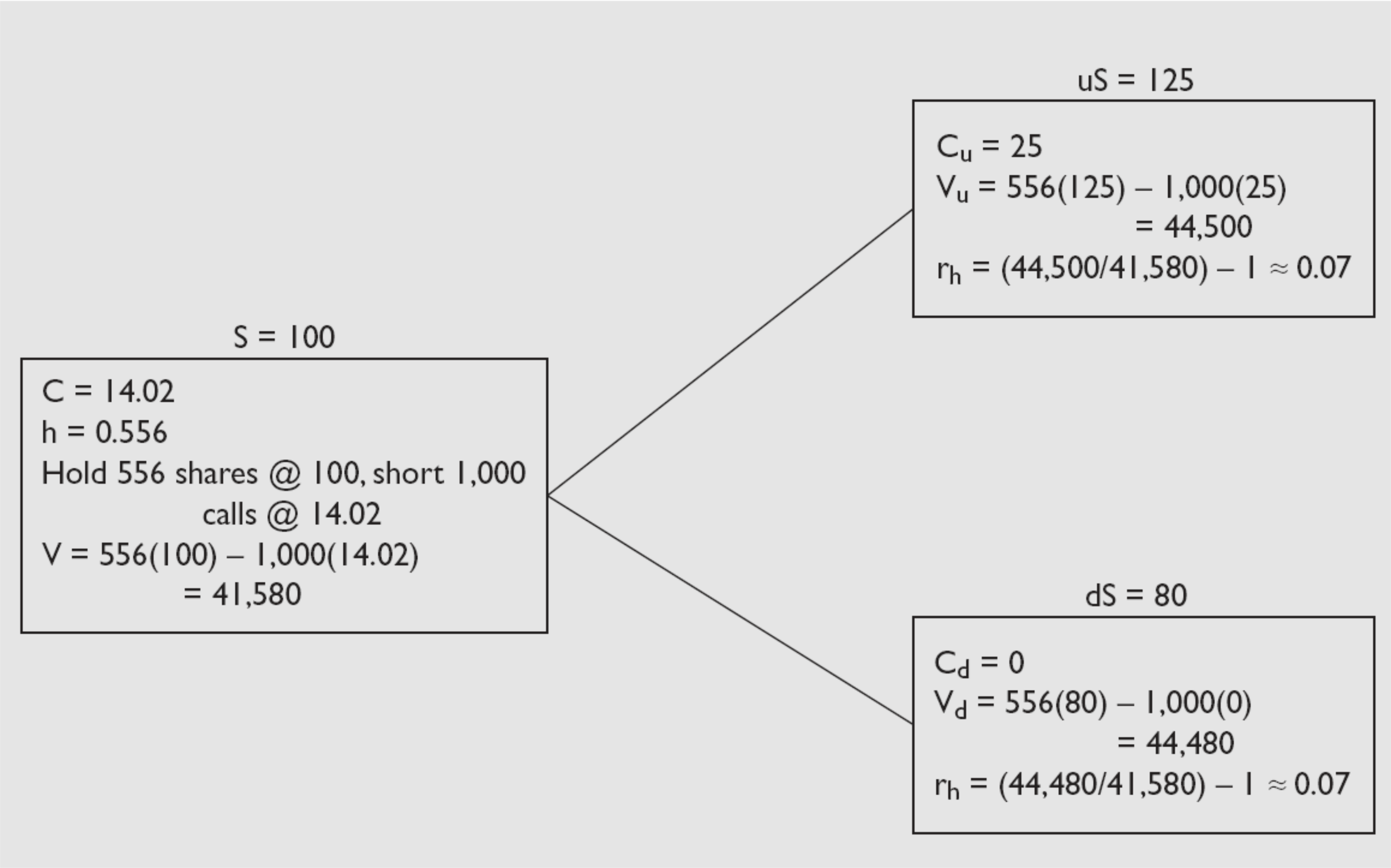

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FIGURE 4.3: The Two-Period Binomial Model

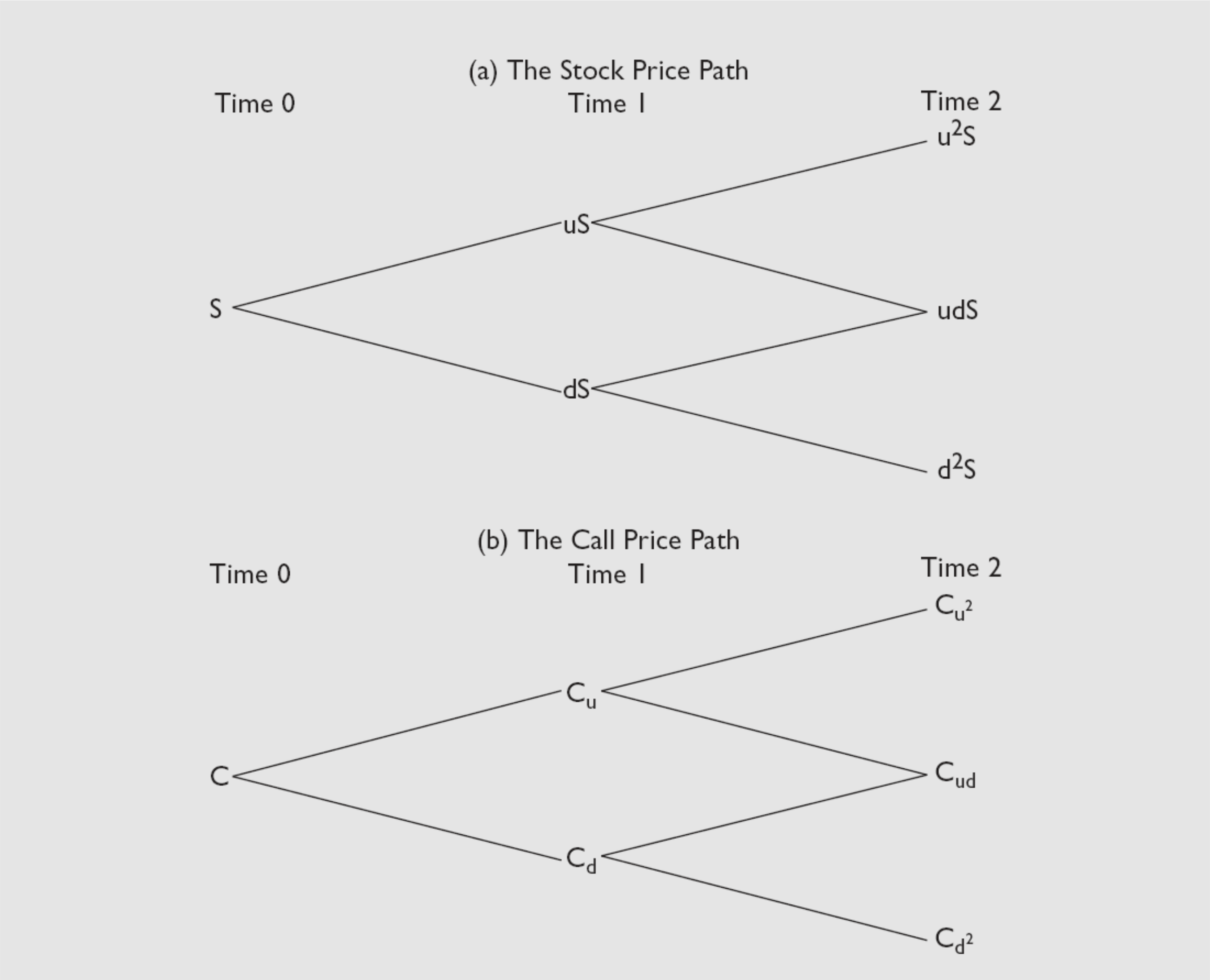

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FIGURE 4.4: Two-Period Binomial Example

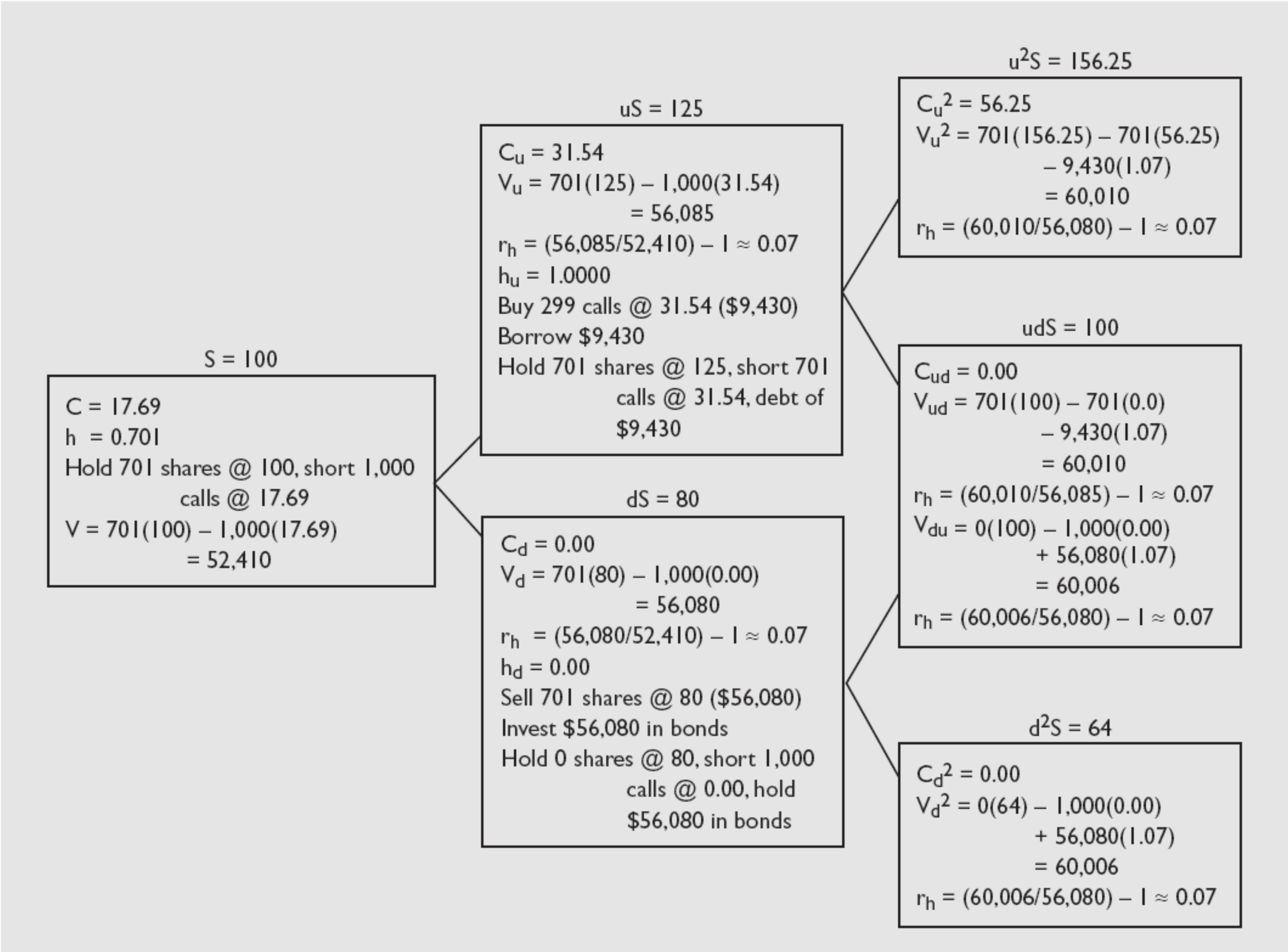

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TABLE 4.1: HEDGE RESULTS IN THE TWO-PERIOD BINOMIAL MODEL

	RETURN FROM HEDGE COMPARED TO RISK-FREE RATE AT PERIOD 1	RETURN FROM HEDGE COMPARED TO RISK-FREE RATE AT PERIOD 2	RETURN FROM HEDGE COMPARED TO RISK-FREE RATE AT TWO-PERIOD
Options Overpriced at Start of Period 1			
Status of option at start of period 2			
Overpriced	Indeterminate	Better	Better
Correctly priced	Better	Equal	Better
Underpriced	Better	Better	Better

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FIGURE 4.5: Two-Period Stock Price Path with 10 Percent Dividend Yield at Time 1

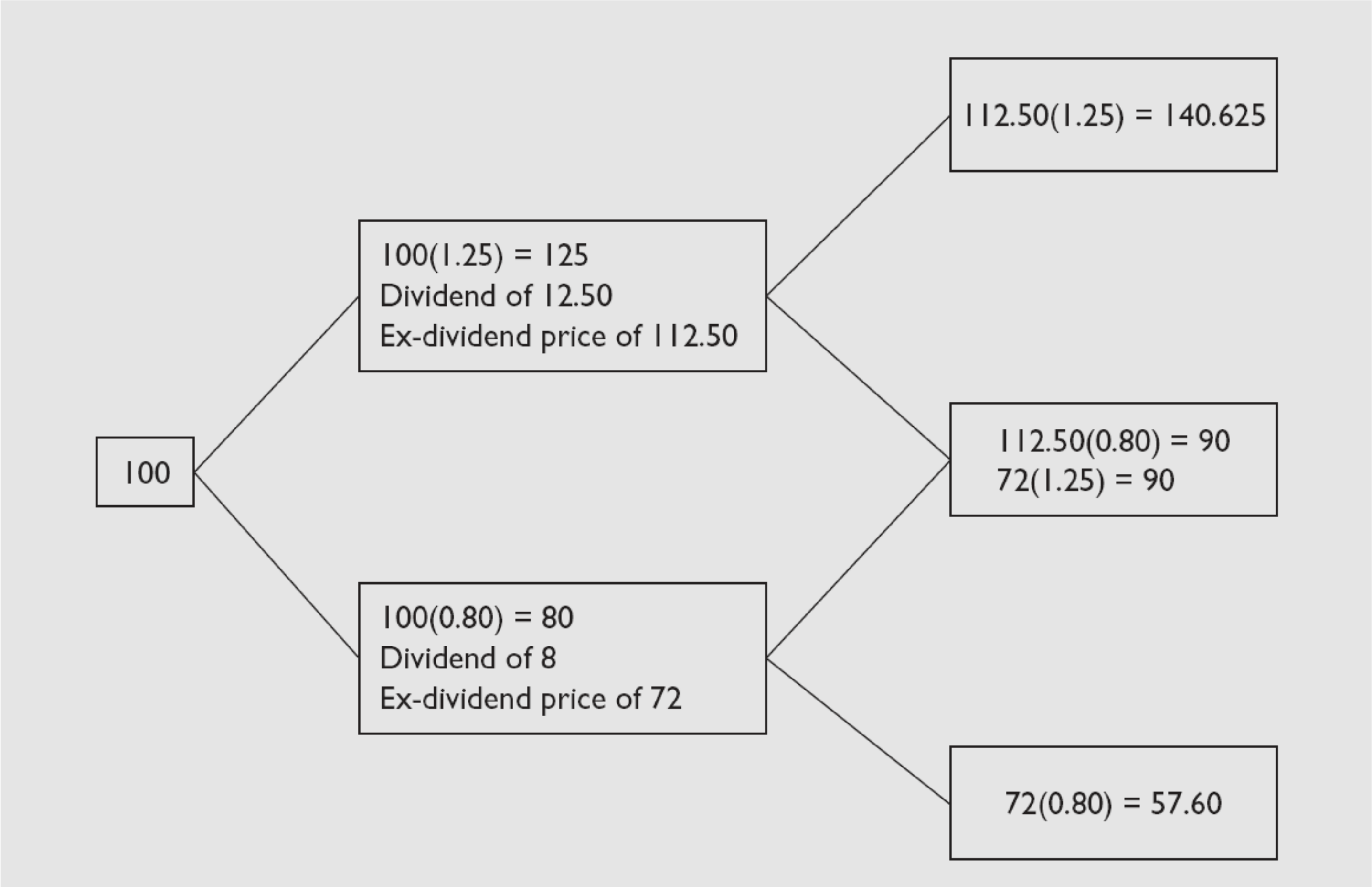

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FIGURE 4.6: Two-Period Stock Price Path with Dividend of $12 at Time 1

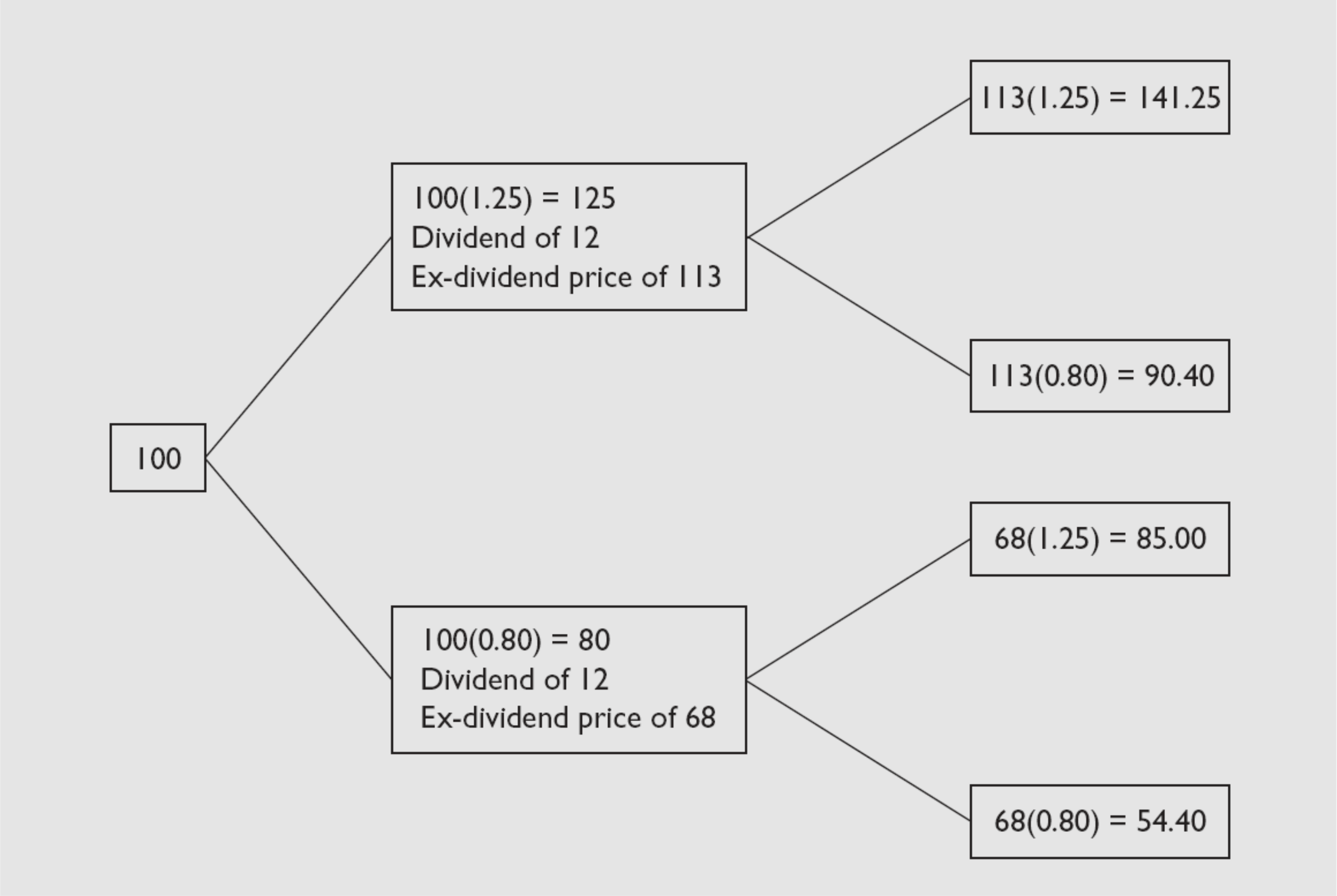

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FIGURE 4.7: Recombining and Non-recombining Three-Period Binomial Trees

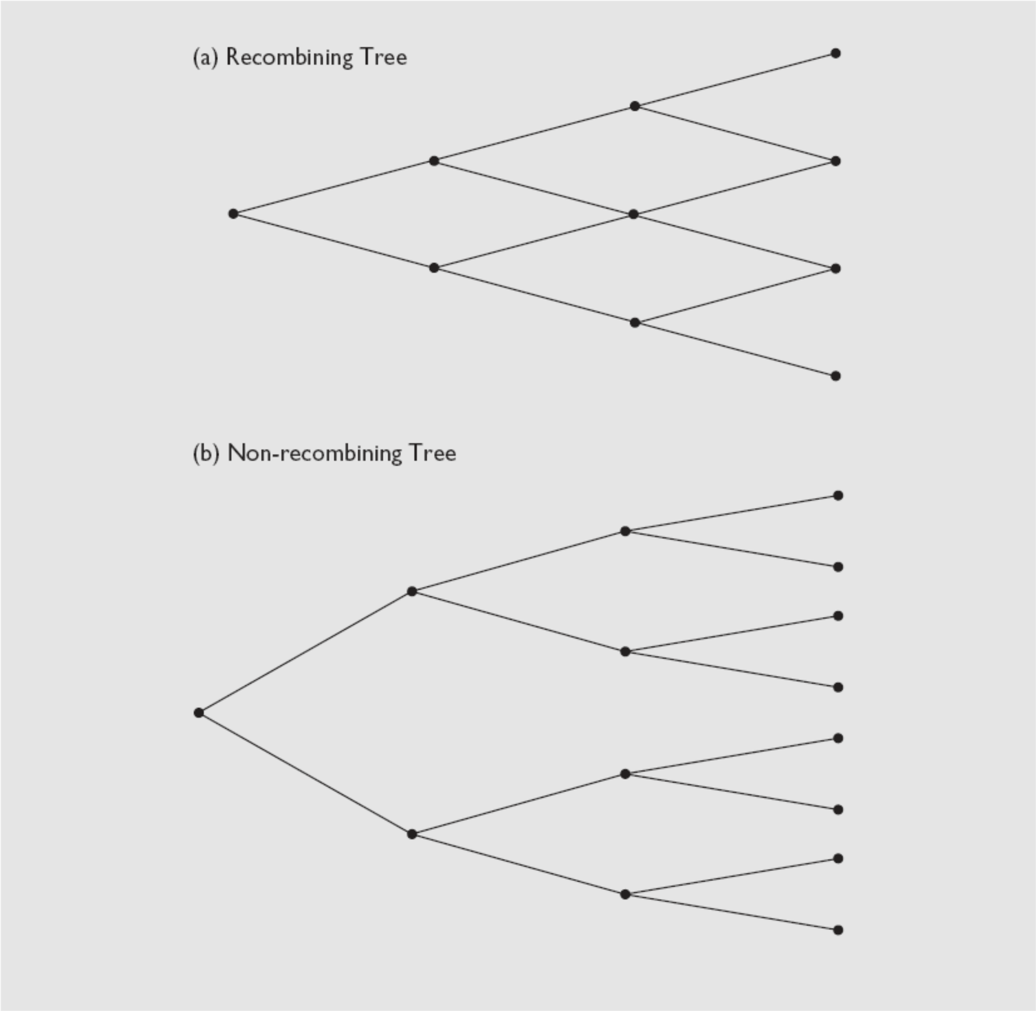

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FIGURE 4.8: Two-Period Stock Price Path with Dividend of $12 at Time 1 and Stock Price Minus Present Value of Dividends Follows the Binomial Process

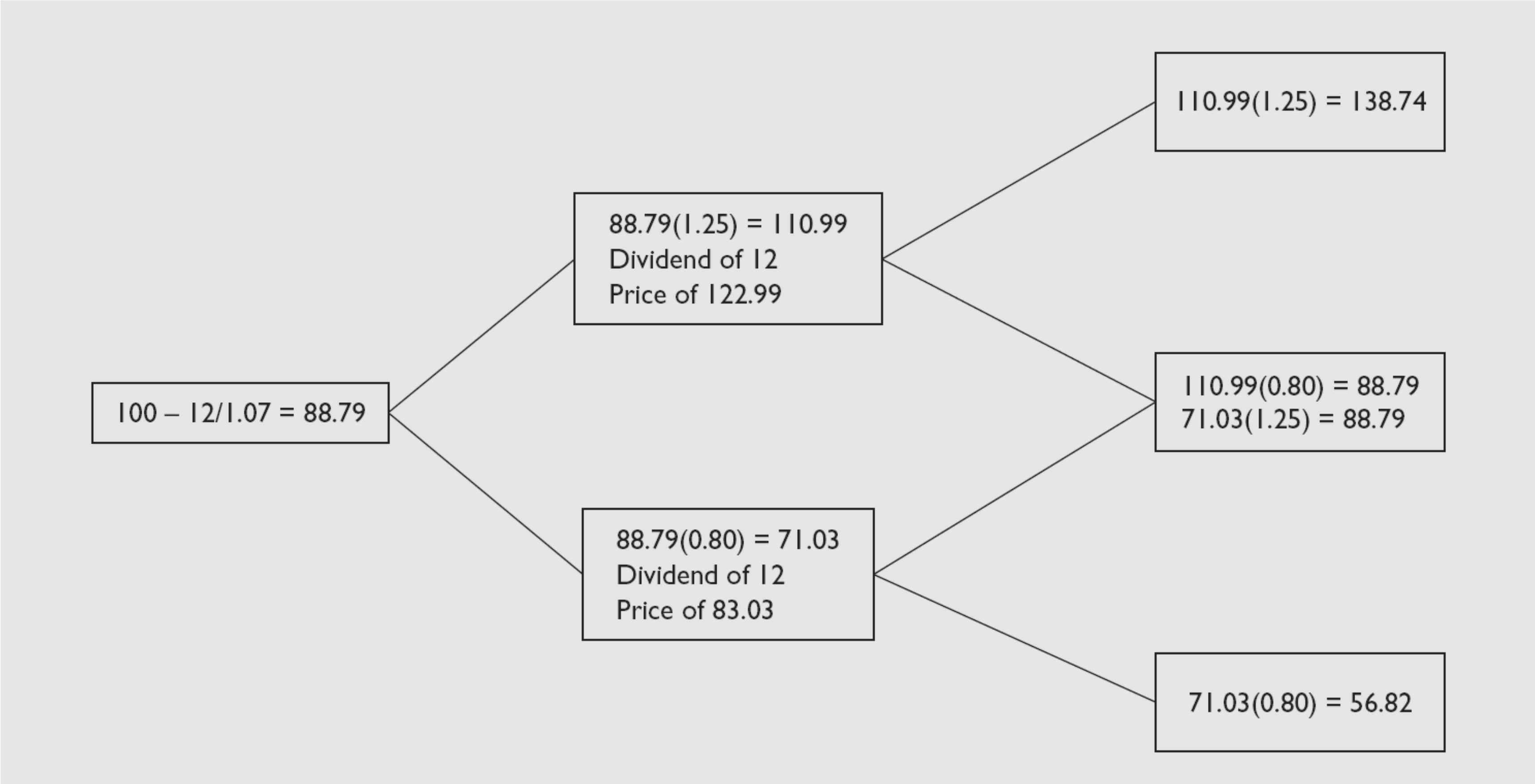

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FIGURE 4.9: Three-Period Binomial Stock Price Path

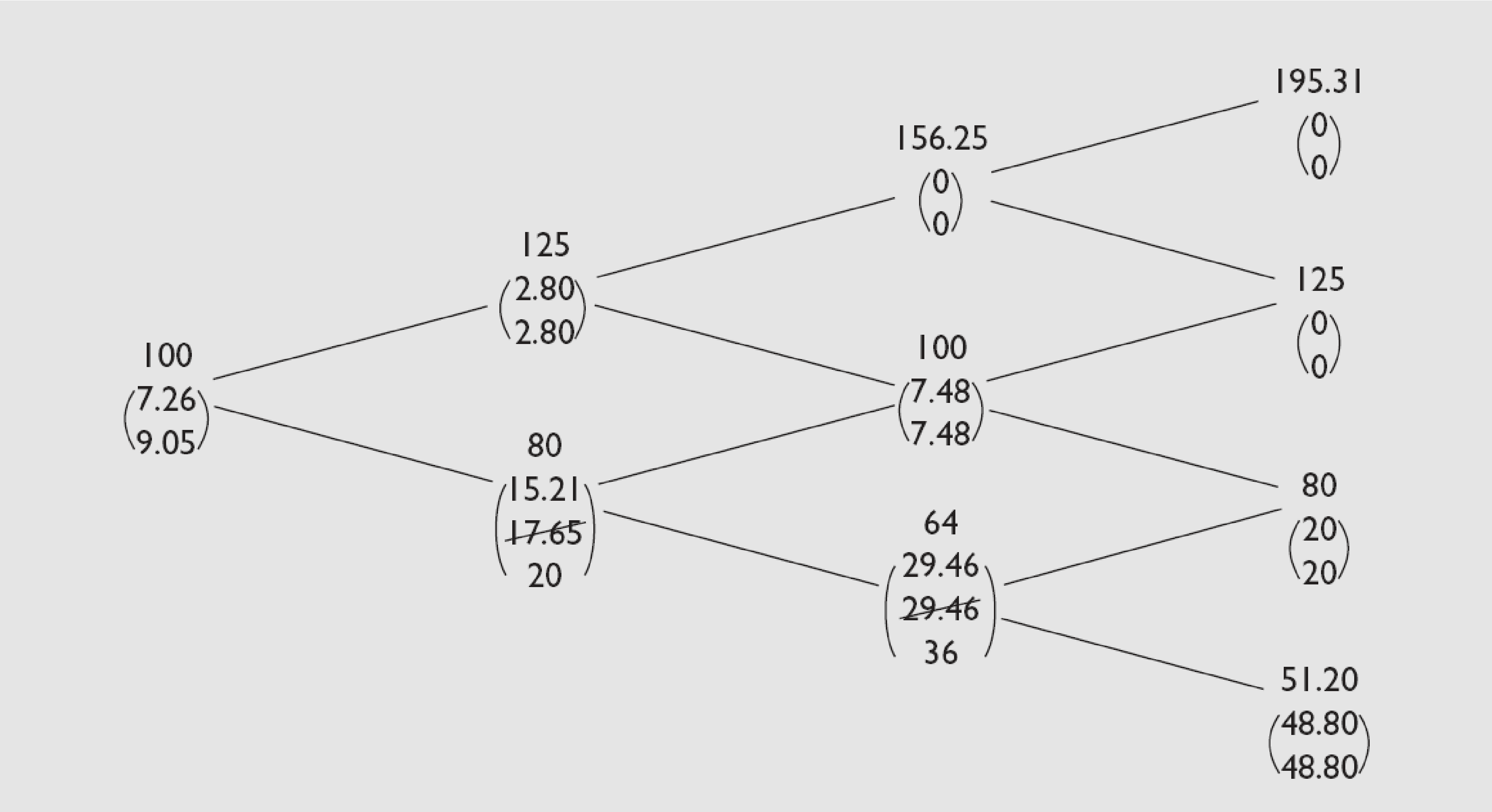

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TABLE 4.2: BINOMIAL OPTION VALUES FOR DCRB JUNE 125 CALL WHEN N IS INCREASED

n	u	d	r	P	C
1	1.2931	0.7733	0.004285	0.4443	16.75
2	1.1993	0.8338	0.002140	0.4605	12.32
5	1.1218	0.8914	0.000856	0.4750	14.19
10	1.0847	0.9219	0.000428	0.4823	13.35
25	1.0528	0.9498	0.000171	0.4888	13.67
50	1.0370	0.9643	0.000086	0.4921	13.54
100	1.0260	0.9747	0.000043	0.4944	13.55
200	1.0183	0.9820	0.000021	0.4960	13.56

Note: S = 125.94, X = 125, r = 4.56 percent, T = 0.0959, $\sigma = 0.83$.

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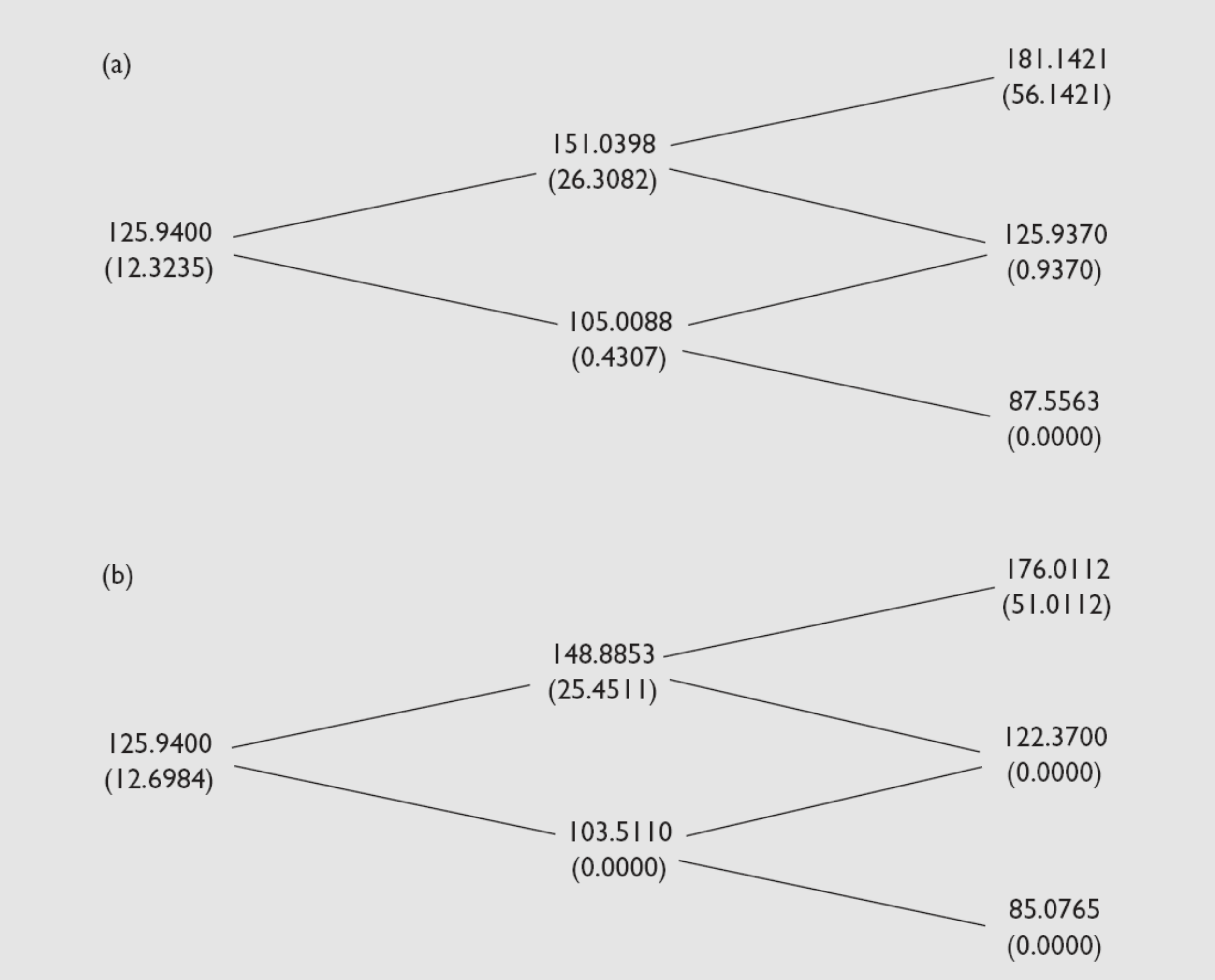
FIGURE 4.10: Pricing the DCRB June 125 Call in a Two-Period World



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