Chapter 5

Option Pricing Models: The Black-Scholes-Merton Model

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Chapter 5: Option Pricing Models: The Black-Scholes-Merton Model

Good theories, like Black-Scholes-Merton, provide a theoretical laboratory in which you can explore the likely effect of possible causes. They give you a common language with which to quantify and communicate your feelings about value.

Emanuel Derman

The Journal of Derivatives, Winter, 2000, p. 64

Important Concepts in Chapter 5 (1 of 2)

- The Black-Scholes-Merton option pricing model
- The relationship of the model’s inputs to the option price
- How to adjust the model to accommodate dividends and put options
- The concepts of historical and implied volatility
- Hedging an option position

Origins of the Black-Scholes-Merton Formula

- Brownian motion and the works of Einstein, Bachelier, Wiener, Itô
- Black, Scholes, Merton and the 1997 Nobel Prize

Black-Scholes-Merton Model as the Limit of the Binomial Model

- Recall the binomial model and the notion of a dynamic risk-free hedge in which no arbitrage opportunities are available.
- Consider the DCRB June 125 call option. Figure 5.1 on 50th slide, shows the model price for an increasing number of time steps.
- The binomial model is in discrete time. As you decrease the length of each time step, it converges to continuous time.

Assumptions of the Model

- Stock prices behave randomly and evolve according to a lognormal distribution.
  - See Figure 5.2a on 51th slide, 5.2b on 52nd slide and 5.3 on 53rd slide for a look at the notion of randomness.
  - A lognormal distribution means that the log (continuously compounded) return is normally distributed. See Figure 5.4 on 54th slide.
- The risk-free rate and volatility of the log return on the stock are constant throughout the option's life
- There are no taxes or transaction costs
- The stock pays no dividends
- The options are European

A Nobel Formula (1 of 12)

- The Black-Scholes-Merton model gives the correct formula for a European call under these assumptions.
- The model is derived with complex mathematics but is easily understandable. The formula is

$$C = S_0 N(d_1) - Xe^{-r_c T} N(d_2)$$

where

$$d_1 = \frac{\ln(S_0 / X) + (r_c + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

A Nobel Formula (2 of 12)

- where
  - $N(d_1)$, $N(d_2)$ = cumulative normal probability
  - σ = annualized standard deviation (volatility) of the continuously compounded return on the stock
  - r_c = continuously compounded risk-free rate

A Nobel Formula (3 of 12)

- A Digression on Using the Normal Distribution
  - The familiar normal, bell-shaped curve
    (Figure 5.5 on 55th slide)
  - See Table 5.1 on 56th slide, for determining the normal probability for d_1 and d_2. This gives you $N(d_1)$ and $N(d_2)$.

A Nobel Formula (4 of 12)

- A Numerical Example
  - Price the DCRB June 125 call
  - S_0 = 125.94, X = 125, r_c = ln(1.0456) = 0.0446, T = 0.0959, σ = 0.83.
  - See Table 5.2 on 57th slide, for calculations. C = $13.21.
  - Familiarize yourself with the accompanying software
    - BlackScholesMertonBinomial10e.xlsm. Note the use of Excel's = normsdist() function.
    - BlackScholesMertonImpliedVolatility10e.xlsm. See Appendix.

A Nobel Formula (5 of 12)

- Characteristics of the Black-Scholes-Merton Formula
  - Interpretation of the Formula
    - The concept of risk neutrality, risk neutral probability, and its role in pricing options
    - The option price is the discounted expected payoff, Max(0, $S_T - X$). We need the expected value of $S_T - X$ for those cases where $S_T > X$.

A Nobel Formula (6 of 12)

- Characteristics of the Black-Scholes-Merton Formula (continued)
  - Interpretation of the Formula (continued)
    - The first term of the formula is the expected value of the stock price given that it exceeds the exercise price times the probability of the stock price exceeding the exercise price, discounted to the present.
    - The second term is the expected value of the payment of the exercise price at expiration.

A Nobel Formula (7 of 12)

- Characteristics of the Black-Scholes-Merton Formula (continued)
  - The Black-Scholes-Merton Formula and the Lower Bound of a European Call
    - Recall from Chapter 3 that the lower bound would be

$$\text{Max}(0, S_0 - Xe^{-r_c T})$$

    - The Black-Scholes-Merton formula always exceeds this value as seen by letting S_0 be very high and then let it approach zero.

A Nobel Formula (8 of 12)

- Characteristics of the Black-Scholes-Merton Formula (continued)
  - The Formula When T = 0
    - At expiration, the formula must converge to the intrinsic value.
    - It does but requires taking limits since otherwise it would be division by zero.
    - Must consider the separate cases of $S_T \geq X$ and $S_T < X$.

A Nobel Formula (9 of 12)

- Characteristics of the Black-Scholes-Merton Formula (continued)
  - The Formula When $S_0 = 0$
    - Here the company is bankrupt so the formula must converge to zero.
    - It requires taking the log of zero, but by taking limits we obtain the correct result.

A Nobel Formula (10 of 12)

- Characteristics of the Black-Scholes-Merton Formula (continued)
  - The Formula When $\sigma = 0$
    - Again, this requires dividing by zero, but we can take limits and obtain the right answer
    - If the option is in-the-money as defined by the stock price exceeding the present value of the exercise price, the formula converges to the stock price minus the present value of the exercise price. Otherwise, it converges to zero.

A Nobel Formula (11 of 12)

- Characteristics of the Black-Scholes-Merton Formula (continued)
  - The Formula When X = 0
    - From Chapter 3, the call price should converge to the stock price.
    - Here both $N(d_1)$ and $N(d_2)$ approach 1.0 so by taking limits, the formula converges to S_0.

A Nobel Formula (12 of 12)

- Characteristics of the Black-Scholes-Merton Formula (continued)
  - The Formula When $r_c = 0$
    - A zero interest rate is not a special case and no special result is obtained.

Variables in the Black-Scholes-Merton Model (1 of 10)

- The Stock Price
  - Let S , then C. See Figure 5.6 on 56th slide.
  - This effect is called the delta, which is given by $N(d_1)$.
  - Measures the change in call price over the change in stock price for a very small change in the stock price.
  - Delta ranges from zero to one. See Figure 5.7 on 59th slide, for how delta varies with the stock price.
  - The delta changes throughout the option's life. See Figure 5.8 on 60th slide.

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Variables in the Black-Scholes-Merton Model (2 of 10)

- The Stock Price (continued)
  - Delta hedging/delta neutral: holding shares of stock and selling calls to maintain a risk-free position
    - The number of shares held per option sold is the delta, $N(d_1)$.
    - As the stock goes up/down by $1, the option goes up/down by $N(d_1)$. By holding $N(d_1)$ shares per call, the effects offset.
    - The position must be adjusted as the delta changes.

Variables in the Black-Scholes-Merton Model (3 of 10)

- The Stock Price (continued)
  - Delta hedging works only for small stock price changes. For larger changes, the delta does not accurately reflect the option price change. This risk is captured by the gamma:

$$\text{Call Gamma} = \frac{e^{-d_1^2/2}}{S_0\sigma\sqrt{2\pi T}}$$

  - For our DCRB June 125 call,

$$\text{Call Gamma} = \frac{e^{-(0.1742)^2/2}}{125.94(0.83)\sqrt{2(3.14159)0.0959}} = 0.0123$$

Variables in the Black-Scholes-Merton Model

- The Stock Price (continued)
  - If the stock goes from 125.94 to 130, the delta is predicted to change from 0.569 to 0.569 + (130 − 125.94)(0.0123) = 0.6189. The actual delta at a price of 130 is 0.6171. So gamma captures most of the change in delta.
  - The larger is the gamma, the more sensitive is the option price to large stock price moves, the more sensitive is the delta, and the faster the delta changes. This makes it more difficult to hedge.
  - See Figure 5.9 on 61st slide, for gamma versus the stock price
  - See Figure 5.10 on 62nd slide, for gamma versus time

Variables in the Black-Scholes-Merton Model (5 of 10)

- The Exercise Price
  - Let X , then C ↓
  - The exercise price does not change in most options so this is useful only for comparing options differing only by a small change in the exercise price.

Variables in the Black-Scholes-Merton Model

- The Risk-Free Rate
  - Take ln(1 + discrete risk-free rate from Chapter 3).
  - Let r_c , then C . See Figure 5.11 on 63rd slide. The effect is called rho

$$\text{Call Rho} = TXe^{-r_c T}N(d_2)$$

  - In our example,

$$\text{Call Rho} = (0.0959)125e^{-0.0446(0.0959)}(0.4670) = 5.57$$

  - If the risk-free rate goes to 0.12, the rho estimates that the call price will go to (0.12 − 0.0446)(5.57) = 0.42. The actual change is 0.43.
  - See Figure 5.12 on 64th slide, for rho versus stock price.

Variables in the Black-Scholes-Merton Model (7 of 10)

- The Volatility or Standard Deviation
  - The most critical variable in the Black-Scholes-Merton model because the option price is very sensitive to the volatility and it is the only unobservable variable.
  - Let σ , then C . See Figure 5.13 on 65th slide.
  - This effect is known as vega.

$$\text{Call vega} = \frac{S_0\sqrt{T}e^{-d_1^2/2}}{\sqrt{2\pi}}$$

  - In our problem this is

$$\text{Call vega} = \frac{125.94\sqrt{0.0959}e^{-0.1742^2/2}}{\sqrt{2(3.14159)}} = 15.32$$

Variables in the Black-Scholes-Merton Model (8 of 10)

- The Volatility or Standard Deviation (continued)
  - Thus if volatility changes by 0.01, the call price is estimated to change by 15.32(0.01) = 0.15
  - If we increase volatility to, say, 0.95, the estimated change would be 15.32(0.12) = 1.84. The actual call price at a volatility of 0.95 would be 15.39, which is an increase of 1.84. The accuracy is due to the near linearity of the call price with respect to the volatility.
  - See Figure 5.14 on 66th slide, for the vega versus the stock price. Notice how it is highest when the call is approximately at-the-money.

Variables in the Black-Scholes-Merton Model (9 of 10)

- The Time to Expiration
  - Calculated as (days to expiration)/365
  - Let T , then C . See Figure 5.15 on 67th slide. This effect is known as theta:

$$\text{Call theta} = -\frac{S_0 \sigma e^{-d_1^2/2}}{2\sqrt{2\pi T}} - r_c X e^{-r_c T} N(d_2)$$

  - In our problem, this would be

$$\text{Call theta} = -\frac{125.94(0.83)e^{-(0.1742)^2/2}}{2\sqrt{2(3.14159)(0.0959)}}$$

$$-(0.0446)125e^{-0.0446(0.0959)}(0.4670) = -68.91$$

Variables in the Black-Scholes-Merton Model (10 of 10)

- The Time to Expiration (continued)
  - If one week elapsed, the call price would be expected to change to (0.0959 − 0.0767)(−68.91) = −1.32. The actual call price with T = 0.0767 is 12.16, a decrease of 1.39.
  - See Figure 5.16 on 68th slide, for theta versus the stock price
  - Note that your spreadsheet BlackScholesMertonBinomial10e.xlsm calculate the delta, gamma, vega, theta, and rho for calls and puts.

Black-Scholes-Merton Model When the Stock Pays Dividends (1 of 2)

- Known Discrete Dividends
  - Assume a single dividend of D_t where the ex-dividend date is time t during the option's life.
  - Subtract present value of dividends from stock price.
  - Adjusted stock price, S', is inserted into the B-S-M model:

$$S'_0 = S_0 - D_t e^{-r_c t}$$

  - See Table 5.3 on 69th slide, for example.
  - The Excel spreadsheet BlackScholesMertonBinomial10e.xlsm allows up to 50 discrete dividends.

Black-Scholes-Merton Model When the Stock Pays Dividends (2 of 2)

- Continuous Dividend Yield
  - Assume the stock pays dividends continuously at the rate of δ.
  - Subtract present value of dividends from stock price. Adjusted stock price, S', is inserted into the B-S-M model:

$$S_0' = S_0 e^{-\delta_c T}$$

  - See Table 5.4 on 71st slide, for example.
  - This approach could also be used if the underlying is a foreign currency, where the yield is replaced by the continuously compounded foreign risk-free rate.
  - The Excel spreadsheet BlackScholesMertonBinomial10e.xlsm permit you to enter a continuous dividend yield.

Black-Scholes-Merton Model and Some Insights into American Call Options

- Table 5.5 on 73rd slide, illustrates how the early exercise decision is made when the dividend is the only one during the option's life
- The value obtained upon exercise is compared to the ex-dividend value of the option.
- High dividends and low time value lead to early exercise.
- Your Excel spreadsheet BlackScholesMertonBinomial10e.xlsm will calculate the American call price using the binomial model.

Estimating the Volatility (1 of 4)

- Historical Volatility
  - This is the volatility over a recent time period.
  - Collect daily, weekly, or monthly returns on the stock.
  - Convert each return to its continuously compounded equivalent by taking ln(1 + return). Calculate variance.
  - Annualize by multiplying by 250 (daily returns), 52 (weekly returns) or 12 (monthly returns). Take square root. See Table 5.6 on 74th slide, for example with DCRB.
  - Your Excel spreadsheet HistoricalVolatility10e.xlsm will do these calculations. See Software Demonstration 5.2.

Estimating the Volatility (2 of 4)

- Implied Volatility
  - This is the volatility implied when the market price of the option is set to the model price.
  - Figure 5.17 on 75th slide, illustrates the procedure.
  - Substitute estimates of the volatility into the B-S-M formula until the market price converges to the model price. See Table 5.7 on 76th slide, for the implied volatilities of the DCRB calls.
  - A short-cut for at-the-money options is

$$\sigma \approx \frac{C}{(0.398)S_0\sqrt{T}}$$

Estimating the Volatility (3 of 4)

- Implied Volatility (continued)
  - For our DCRB June 125 call, this gives

$$\sigma = \frac{13.50}{(0.398)125.94\sqrt{0.0959}} = 0.8697$$

  - This is quite close; the actual implied volatility is 0.83.
  - Appendix 5.A shows a method to produce faster convergence.

Estimating the Volatility (4 of 4)

- Implied Volatility (continued)
  - Interpreting the Implied Volatility
    - The relationship between the implied volatility and the time to expiration is called the term structure of implied volatility. See Figure 5.18 on 77th slide.
    - The relationship between the implied volatility and the exercise price is called the volatility smile or volatility skew. Figure 5.19 on 78th slide. These volatilities are actually supposed to be the same. This effect is puzzling and has not been adequately explained.
    - The CBOE has constructed indices of implied volatility of one-month at-the-money options based on the S&P 100 (VIX) and Nasdaq (VXN). See Figure 5.20 on 79th slide.

Put Option Pricing Models (1 of 2)

- Restate put-call parity with continuous discounting

$$P_e(S_0,T,X) = C_e(S_0,T,X) - S_0 + Xe^{-r_c T}$$

- Substituting the B-S-M formula for C above gives the B-S-M put option pricing model

$$P = Xe^{-r_c T}[1 - N(d_2)] - S_0[1 - N(d_1)]$$

- $N(d_1)$ and $N(d_2)$ are the same as in the call model.

Put Option Pricing Models (2 of 2)

- Note calculation of put price:

$$P = 125e^{-(0.0446)0.0959}[1-0.4670]$$
$$-125.94[1-0.5692] \quad = \quad 12.08$$

- The Black-Scholes-Merton price does not reflect early exercise and, thus, is extremely biased here since the American option price in the market is 11.50. A binomial model would be necessary to get an accurate price. With n = 100, we obtained 12.11.
- See Table 5.8 on 80th slide, for the effect of the input variables on the Black-Scholes-Merton put formula.
- Your software also calculates put prices and Greeks.

Managing the Risk of Options (1 of 7)

- Here we talk about how option dealers hedge the risk of option positions they take.
- Assume a dealer sells 1,000 DCRB June 125 calls at the Black-Scholes-Merton price of 13.5533 with a delta of 0.5692. Dealer will buy 569 shares and adjust the hedge daily.
  - To buy 569 shares at $125.94 and sell 1,000 calls at $13.5533 will require $58,107.
  - We simulate the daily stock prices for 35 days, at which time the call expires.

Managing the Risk of Options (2 of 7)

- The second day, the stock price is 120.4020. There are now 34 days left. Using BlackScholesMertonBinomial10e.xlsm, we get a call price of 10.4078 and delta of 0.4981. We have
  - Stock worth 569($120.4020) = $68,509
  - Options worth −1,000($10.4078) = −$10,408
  - Total of $58,101
  - Had we invested $58,107 in bonds, we would have had

    $\$58{,}107e^{0.0446(1/365)} = \$58{,}114.$
- Table 5.9 on 83rd slide, shows the remaining outcomes. We must adjust to the new delta of 0.4981. We need 498 shares so sell 71 and invest the money ($8,549) in bonds.

Managing the Risk of Options (3 of 7)

- At the end of the second day, the stock goes to 126.2305 and the call to 13.3358. The bonds accrue to a value of $8,550. We have
  - Stock worth 498($126.2305) = $62,863
  - Options worth −1,000($13.3358) = −$13,336
  - Bonds worth $8,550 (includes one days' interest)
  - Total of $58,077
  - Had we invested the original amount in bonds, we would have had $\$58{,}107e^{0.0446(2/365)} = \$58{,}121$. We are now short by over $44.
- At the end we have $59,762, a excess of $1,406.

Managing the Risk of Options (4 of 7)

- What we have seen is the second order or gamma effect. Large price changes, combined with an inability to trade continuously result in imperfections in the delta hedge.
- To deal with this problem, we must gamma hedge, i.e., reduce the gamma to zero. We can do this only by adding another option. Let us use the June 130 call, selling at 11.3792 with a delta of 0.5087 and gamma of 0.0123. Our original June 125 call has a gamma of 0.0121. The stock gamma is zero.
- We shall use the symbols Δ_1, Δ_2, Γ_1 and Γ_2. We use h_S shares of stock and h_C of the June 130 calls.

Managing the Risk of Options (5 of 7)

- The delta hedge condition is
  - $h_s(1) - 1{,}000\Delta_1 + h_c\Delta_2 = 0$
- The gamma hedge condition is
  - $-1{,}000\Gamma_1 + h_c\Gamma_2 = 0$
- We can solve the second equation and get h_C and then substitute back into the first to get h_S. Solving for h_C and h_S, we obtain
  - $h_c = 1{,}000\ (0.0121/0.0123) = 984$
  - $h_S = 1{,}000\left(0.5692 - (0.0121/0.0123)0.5087\right) = 68$
- So buy 68 shares, sell 1,000 June 125s, buy 984 June 130s.

Managing the Risk of Options (6 of 7)

- The initial outlay will be
  - $68(\$125.94) - 1{,}000(\$13.5533) + 985(\$11.3792) = \$6{,}219$
- At the end of day one, the stock is at 120.4020, the 125 call is at 10.4078, the 130 call is at 8.5729. The portfolio is worth
  - $68(\$120.4020) - 1{,}000(\$10.4078) + 985(\$8.5729) = \$6{,}224$
- It should be worth $\$6{,}218e^{0.0446(1/365)} = \$6{,}220$.
- The new deltas are 0.4981 and 0.4366 and the new gammas are 0.0131 and 0.0129.

Managing the Risk of Options (7 of 7)

- The new values are 1,013 of the 130 calls so we buy 28. The new number of shares is 56 so we sell 12. Overall, this generates $1,444, which we invest in bonds.
- The next day, the stock is at $126.2305, the 125 call is at $13.3358 and the 130 call is at $11.1394. The bonds are worth $1,205. The portfolio is worth
  - 56($126.2305) − 1,000($13.3358) + 1,013($11.1394) + $1,205 = $6,222.
- The portfolio should be worth $\$6,219e^{0.0446(2/365)} = \$6,221$.
- Continuing this, we end up at $6,267 and should have $6,246, a difference of $21. We are much closer than when only delta hedging.

When the Black-Scholes-Merton may or may not hold

- Liquidity
- Short-Selling
- Information Asymmetry
- Problems with Exotic Options
- Performativity and Counter-Performativity

Summary

- See Figure 5.21 on 85th slide, for the relationship between call, put, underlying asset, risk-free bond, put-call parity, and Black-Scholes-Merton call and put option pricing models.

Appendix 5.A: A Shortcut to the Calculation of Implied Volatility (1 of 3)

- This technique developed by Manaster and Koehler gives a starting point and guarantees convergence. Let a given volatility be σ* and the corresponding Black-Scholes-Merton price be C(σ*). The initial guess should be

$$\sigma_1^* = \sqrt{\left|\ln\left(\frac{S_0}{X}\right) + r_c T\right|\left(\frac{2}{T}\right)}$$

- You then compute $C(\sigma_1^*)$. If it is not close enough, you make the next guess.

Appendix 5.A: A Shortcut to the Calculation of Implied Volatility (2 of 3)

- Given the i^{th} guess, the next guess should be

$$\sigma^*_{i+1} = \sigma^*_i - \frac{\left[C(\sigma^*_i) - C(\sigma)\right] e^{d_1^2/2} \sqrt{2\pi}}{S_0 \sqrt{T}}$$

- where d_1 is computed using σ^*_1. Let us illustrate using the DCRB June 125 call. $C(\sigma) = 13.50$. The initial guess is

$$\sigma^*_1 = \sqrt{\left|\ln\left(\frac{125.9375}{125}\right) + 0.0446(0.0959)\right|\left(\frac{2}{0.0959}\right)} = 0.4950$$

Appendix 5.A: A Shortcut to the Calculation of Implied Volatility (3 of 3)

- At a volatility of 0.4950, the Black-Scholes-Merton value is 8.41. The next guess should be

$$\sigma_2^* = 0.4950 - \frac{[8.41 - 13.50]e^{(0.1533)^2/2}(2.5066)}{125.9375\sqrt{0.0959}} = 0.8260$$

- where 0.1533 is d_1 computed from the Black-Scholes-Merton model using 0.4950 as the volatility and 2.5066 is the square root of 2π. Now using 0.8260, we obtain a Black-Scholes-Merton value of 13.49, which is close enough to 13.50. So 0.83 is the implied volatility.

FIGURE 5.1: Binomial Option Prices for Different n; DCRB June 125 Call; S_0 = 125.94, r = 0.0456, T = 0.0959, σ = 0.83

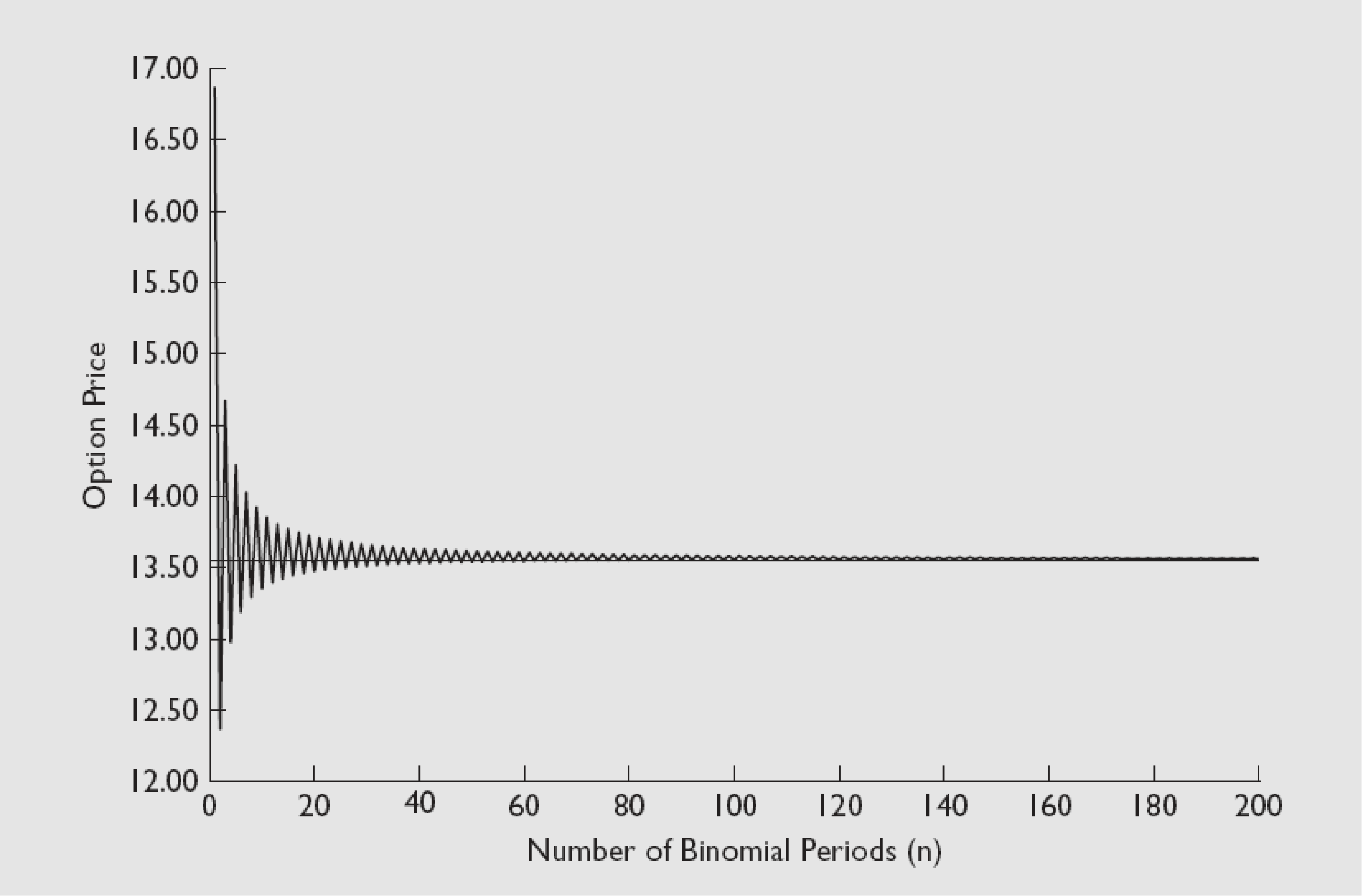

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FIGURE 5.2a: Simulated or Real Stock Prices of DCRB

(a)

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FIGURE 5.2b: Simulated or Real Stock Prices of DCRB

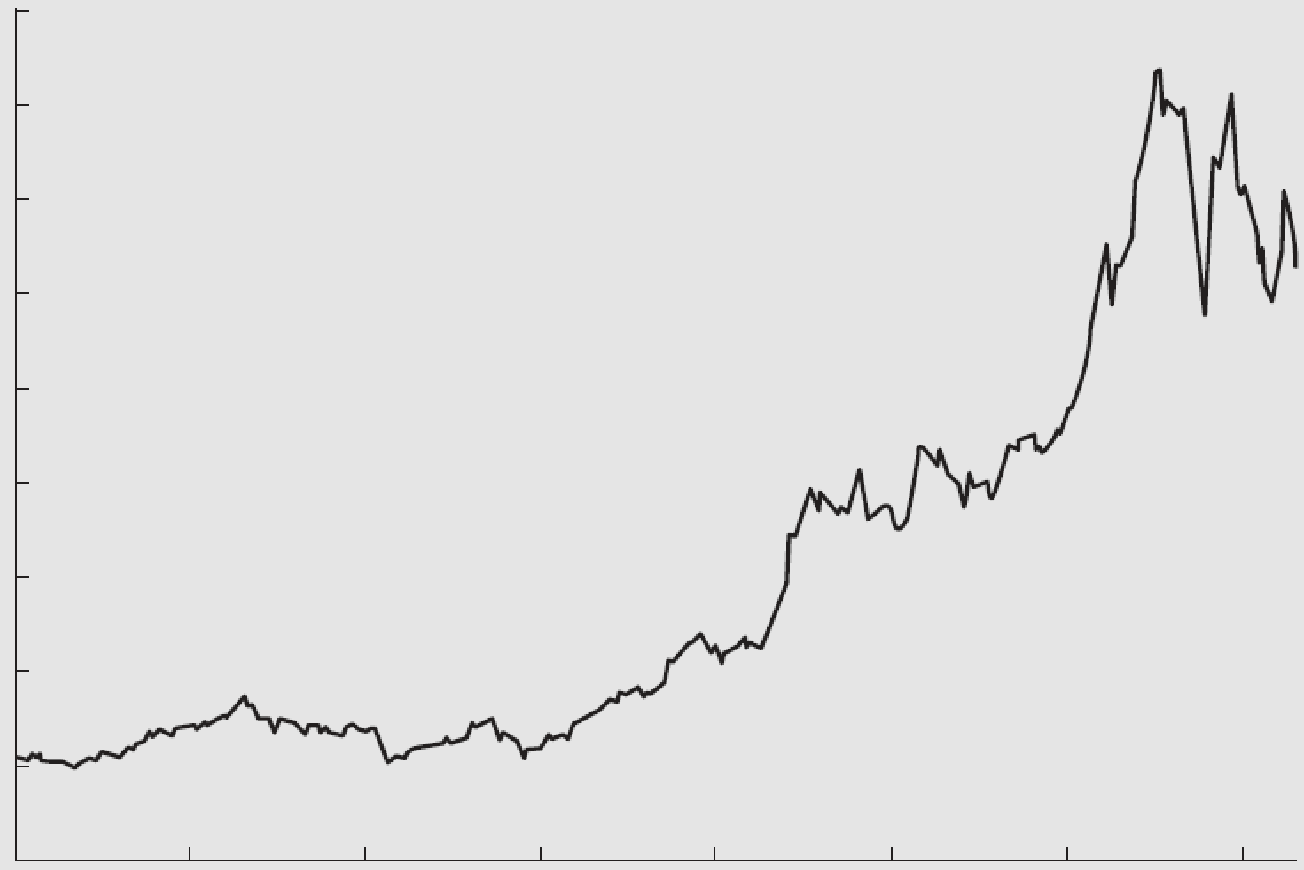
(b)

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FIGURE 5.3: Daily Returns on DCRB

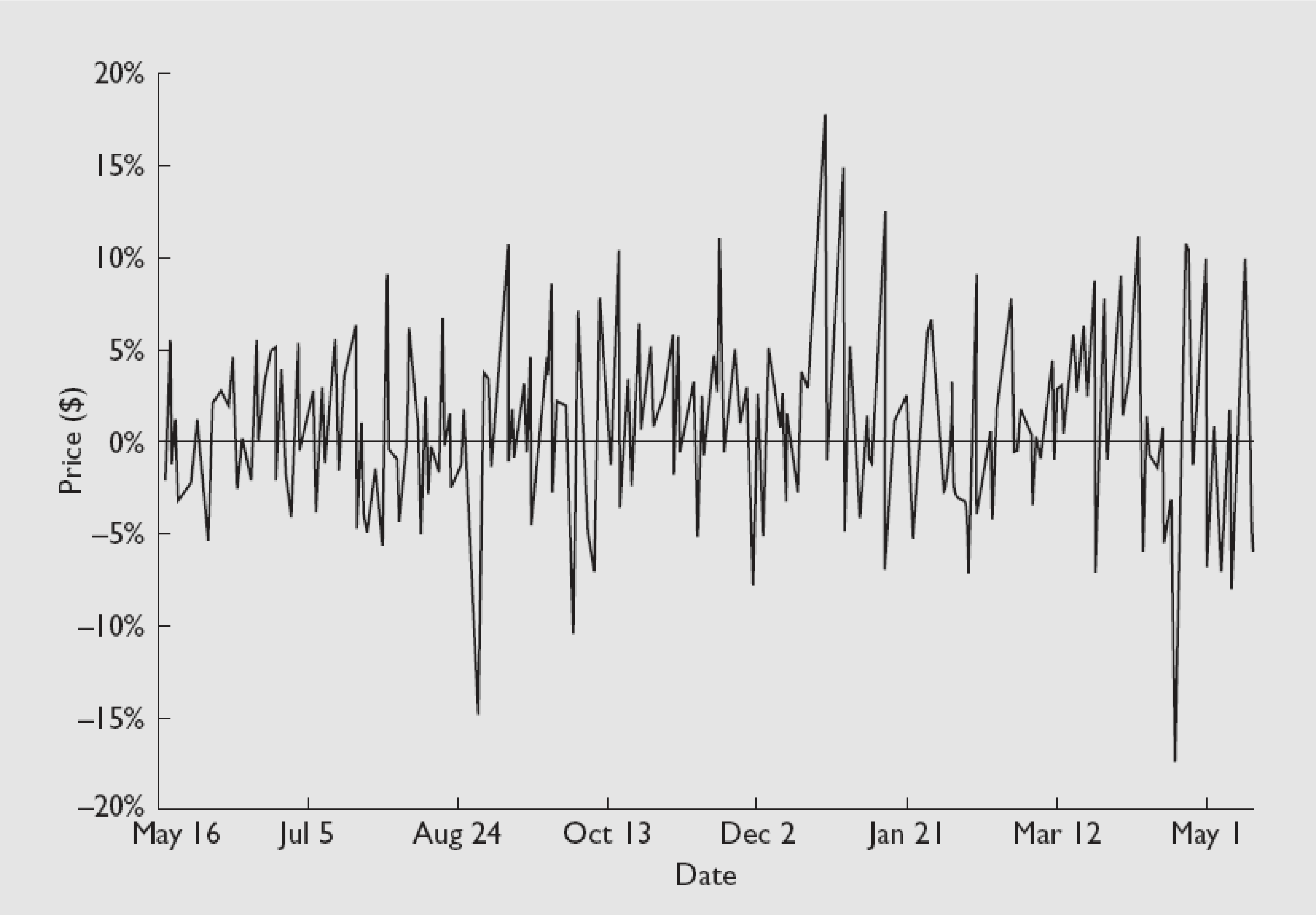

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FIGURE 5.4: Histogram of Daily Log (Continuously Compounded) Returns of DCRB

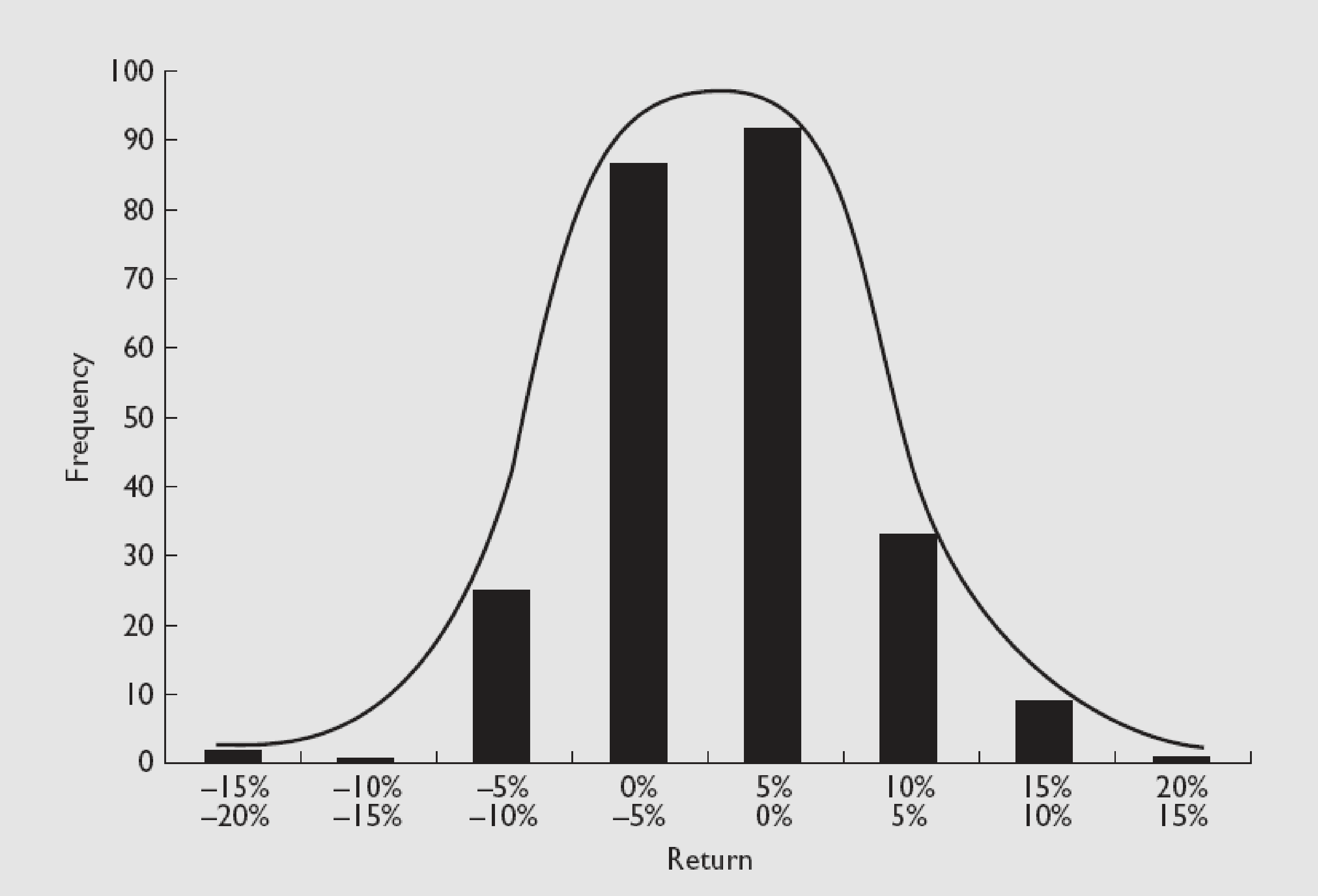

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FIGURE 5.5: The Standard Normal Probability Distribution

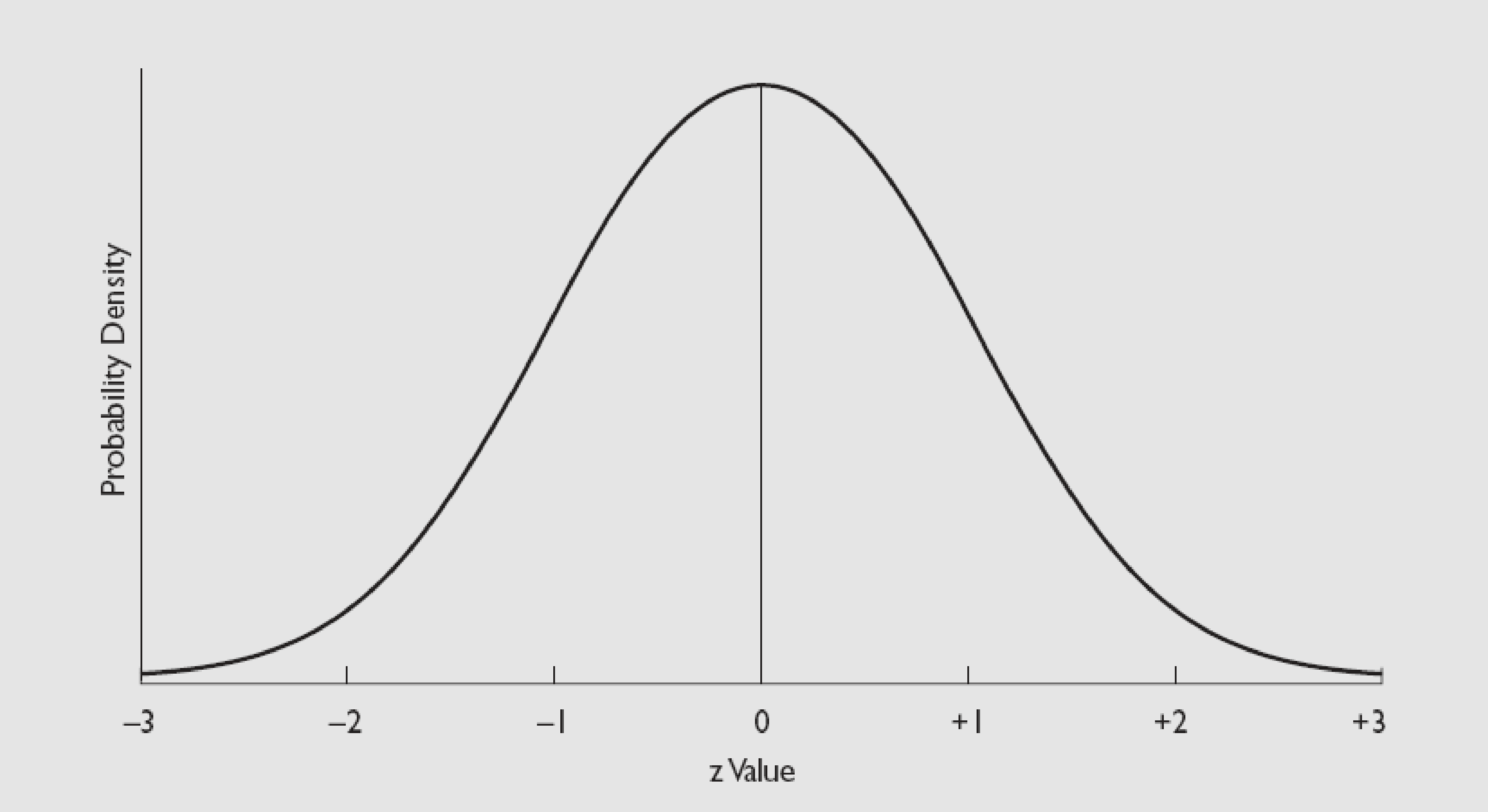

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TABLE 5.1: STANDARD NORMAL PROBABILITIES

Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8860	0.8888	0.8907	0.8925	0.8943	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

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TABLE 5.2: CALCULATING THE BLACK–SCHOLES–MERTON PRICE

$S_0 = 125.94$ $X = 125$ $r_c = 0.0446$ $\sigma = 0.83$ $T = 0.0959$

1. Compute d_1.

$$d_1 = \frac{\ln(125.94/125) + [0.0446 + (0.83)^2/2]0.0959}{0.83\sqrt{0.0959}} = 0.1743$$

2. Compute d_2.

$$d_2 = 0.1743 - 0.83\sqrt{0.0959} = -0.0827$$

3. Look up $N(d_1)$.

$$N(0.17) = 0.5675$$

4. Look up $N(d_2)$.

$$N(-0.08) = 1 - N(0.08) = 1 - 0.5319 = 0.4681$$

5. Plug into formula for C.

$$C = 125.94(0.5675) - 125e^{-0.0446(0.0959)}(0.4681) = 13.21$$

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FIGURE 5.6: The Call Price as a Function of the Stock Price DCRB June 125; $r_c = 0.0446$, $T = 0.0959$, $\sigma = 0.83$

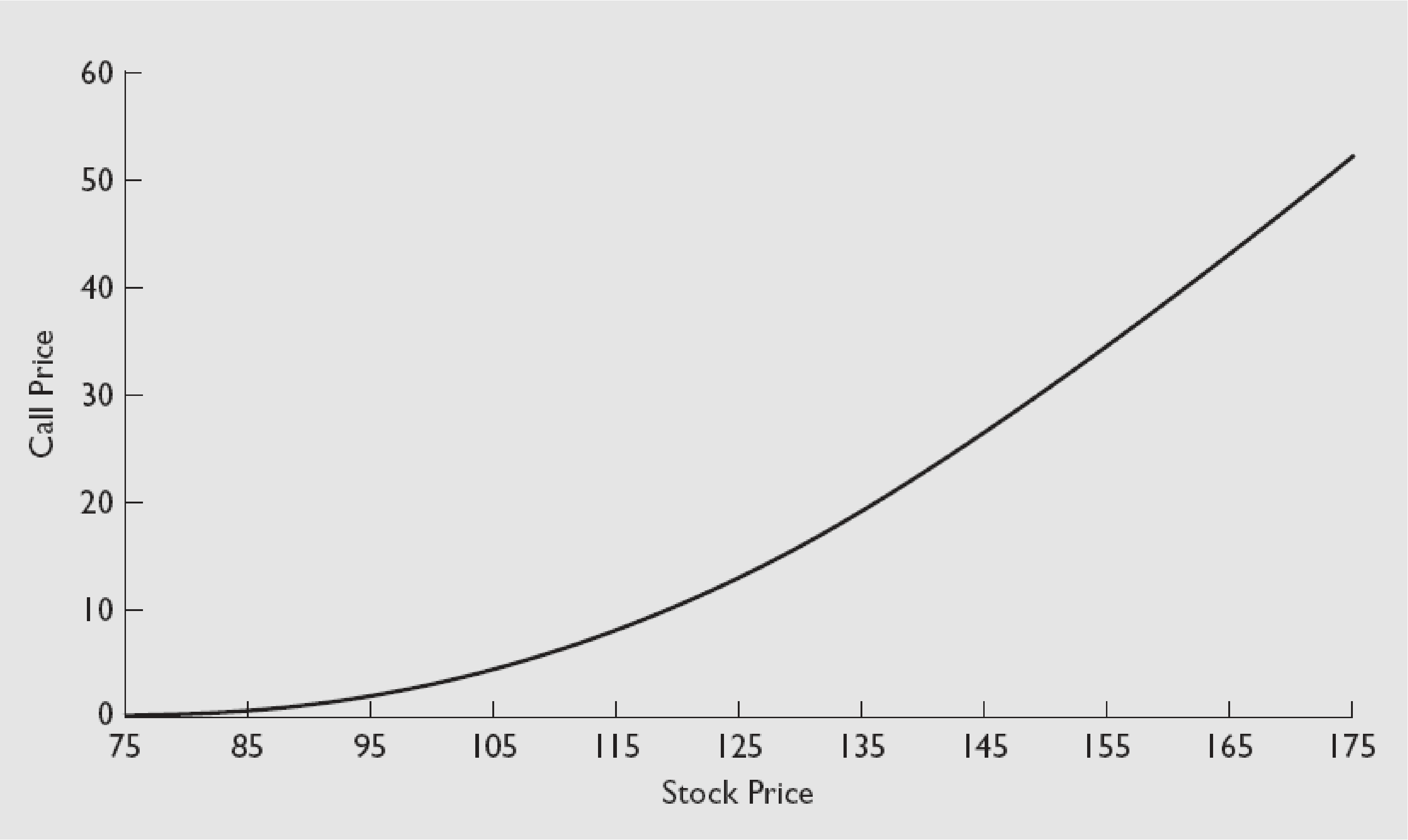

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FIGURE 5.7: The Delta of a Call as a Function of the Stock Price DCRB June 125; S_0 = 125.94, r_c = 0.0446, σ = 0.83

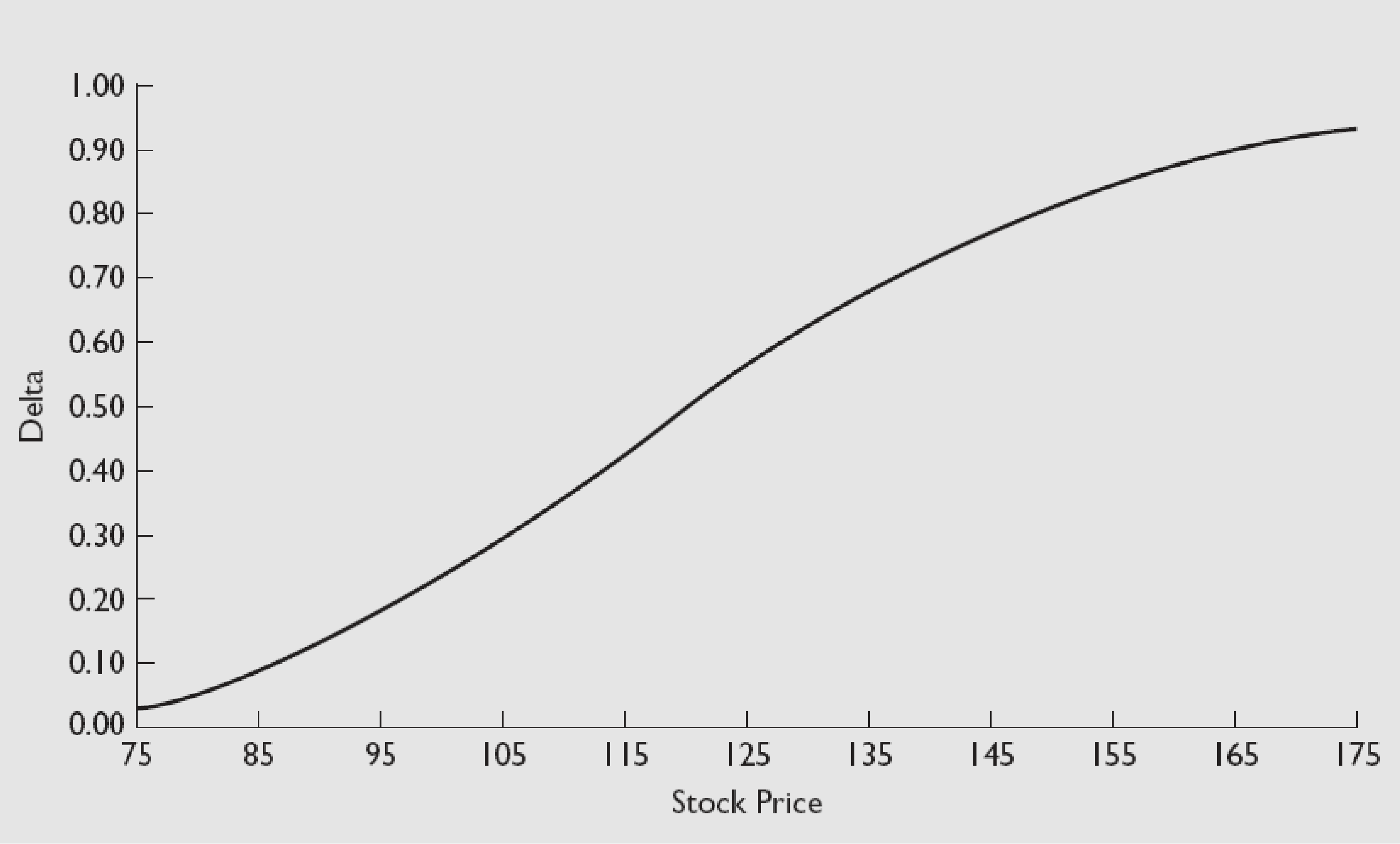

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FIGURE 5.8: The Delta of a Call as a Function of the Time in Days to Expiration DCRB June 125; $S_0 = 125.94$, $r_c = 0.0446$, $\sigma = 0.83$

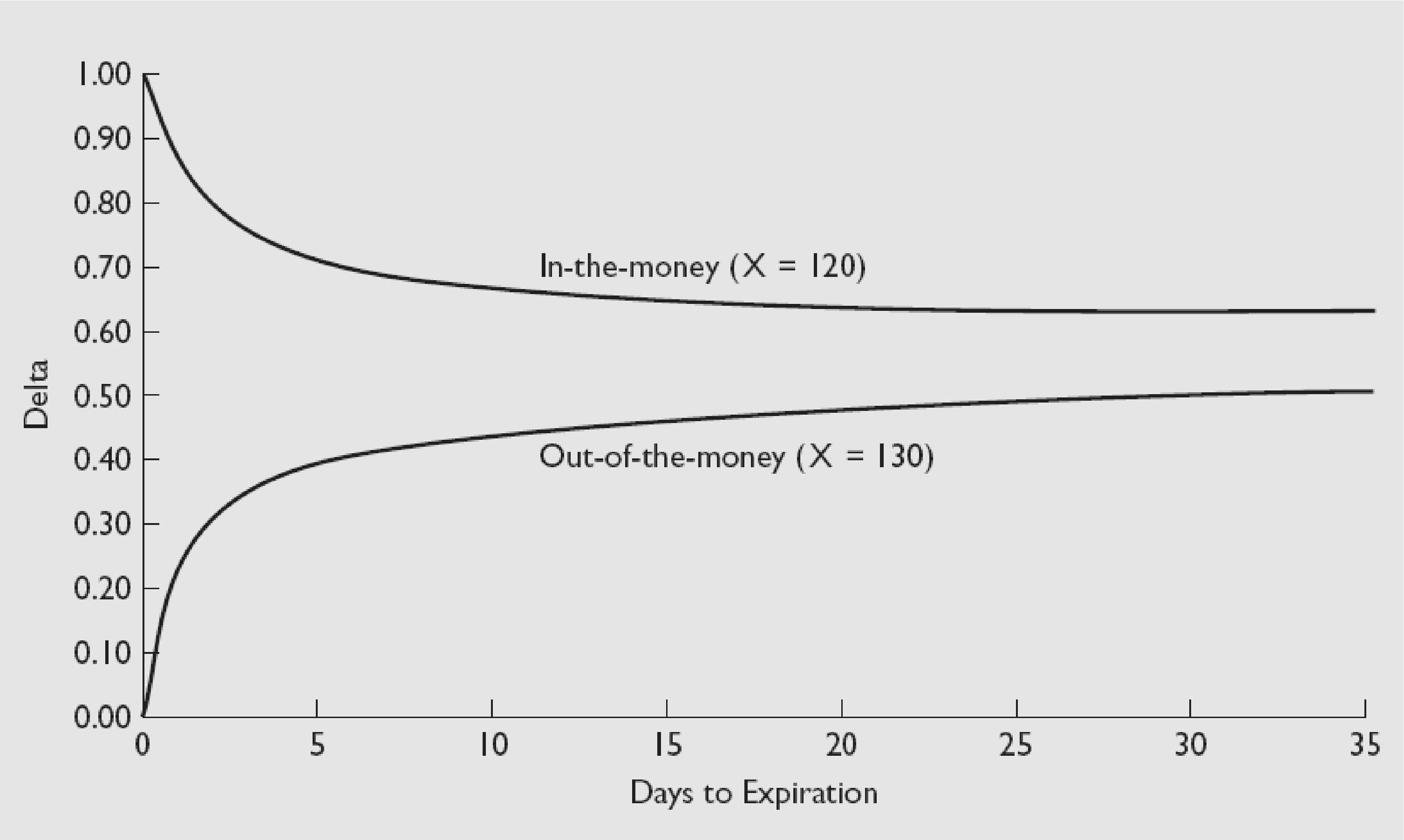

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FIGURE 5.9: The Gamma of a Call Price as a Function of the Stock Price DCRB June 125; $r_c = 0.0446$, $T = 0.0959$, $\sigma = 0.83$

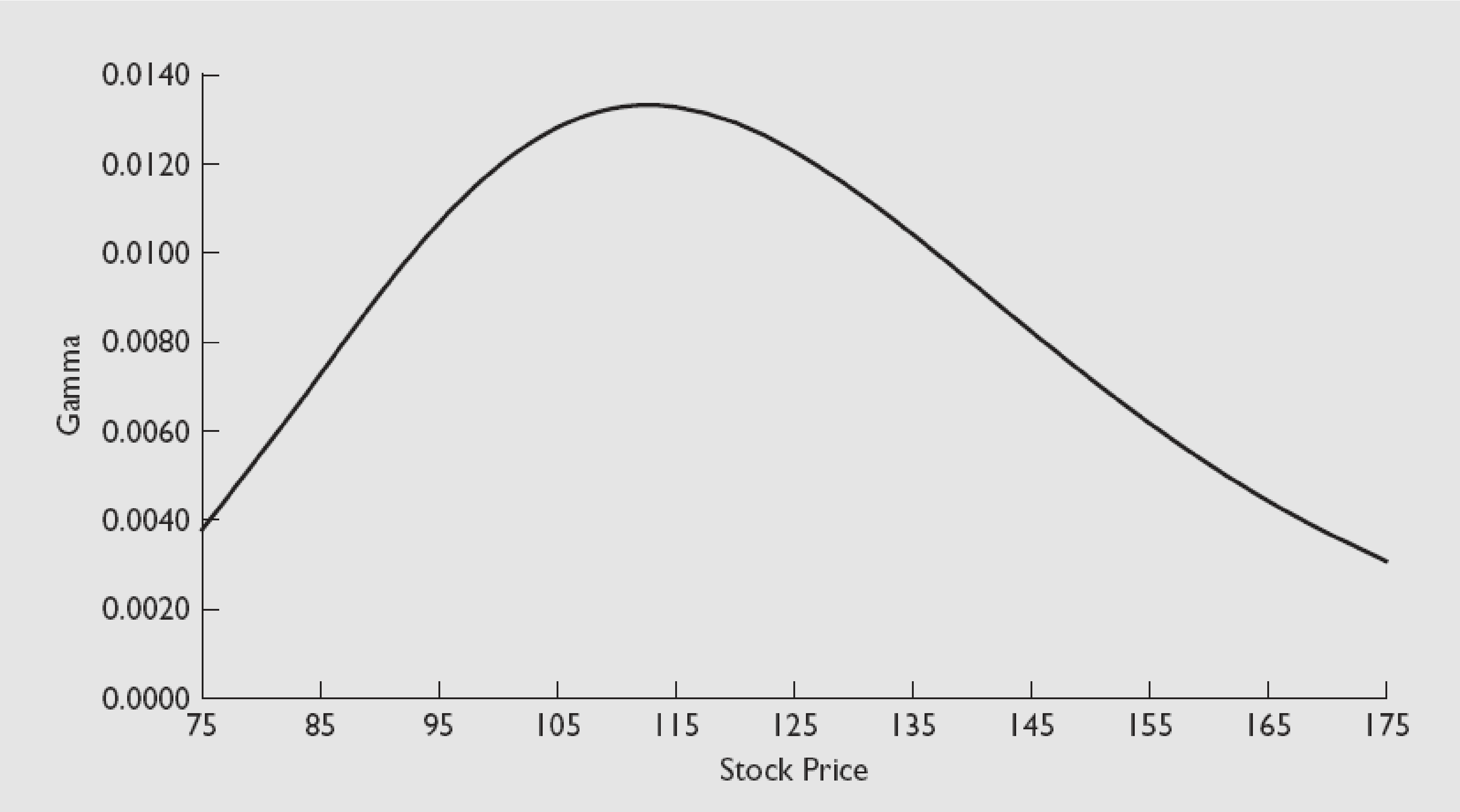

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FIGURE 5.10: The Gamma of a Call Price as a Function of the Time in Days to Expiration DCRB June 125; S_0 = 125.94, rc = 0.0446, σ = 0.83

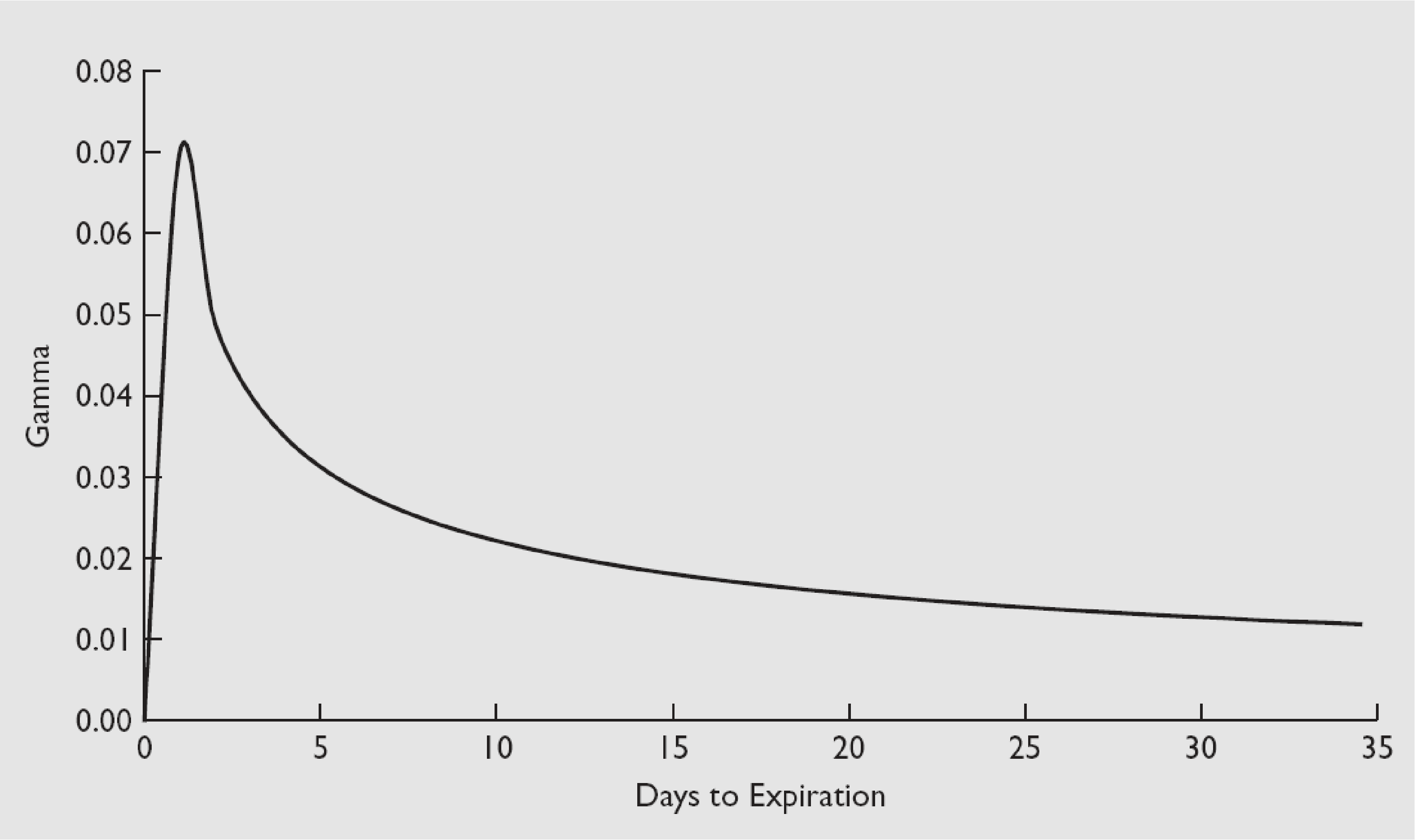

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FIGURE 5.11: The Call Price as a Function of the Risk-Free Rate DCRB June 125; S_0 = 125.94, T = 0.0959, σ = 0.83

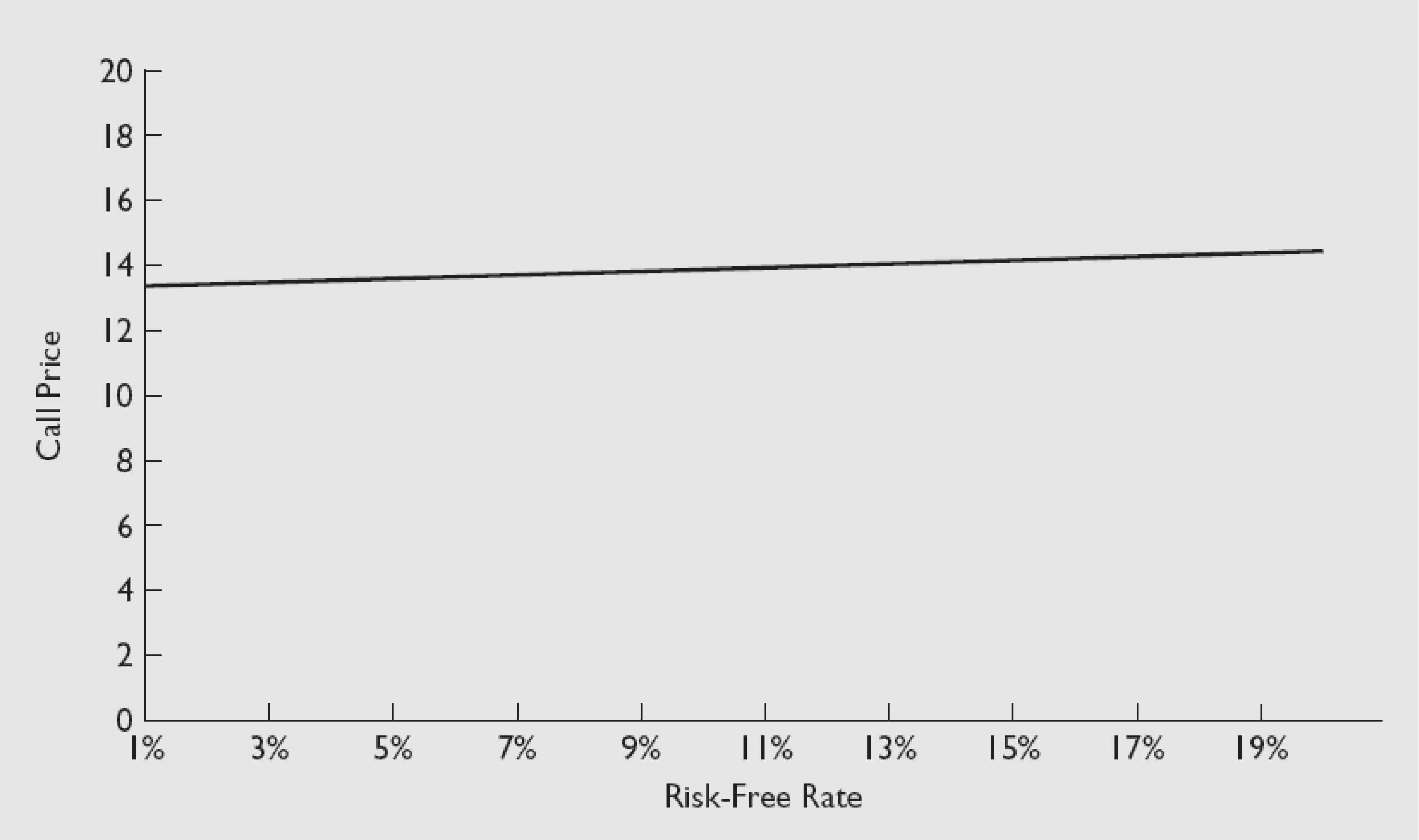

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FIGURE 5.12: The Rho of a Call as a Function of the Stock Price DCRB June 125; $r_c = 0.0446$, $T = 0.0959$, $\sigma = 0.83$

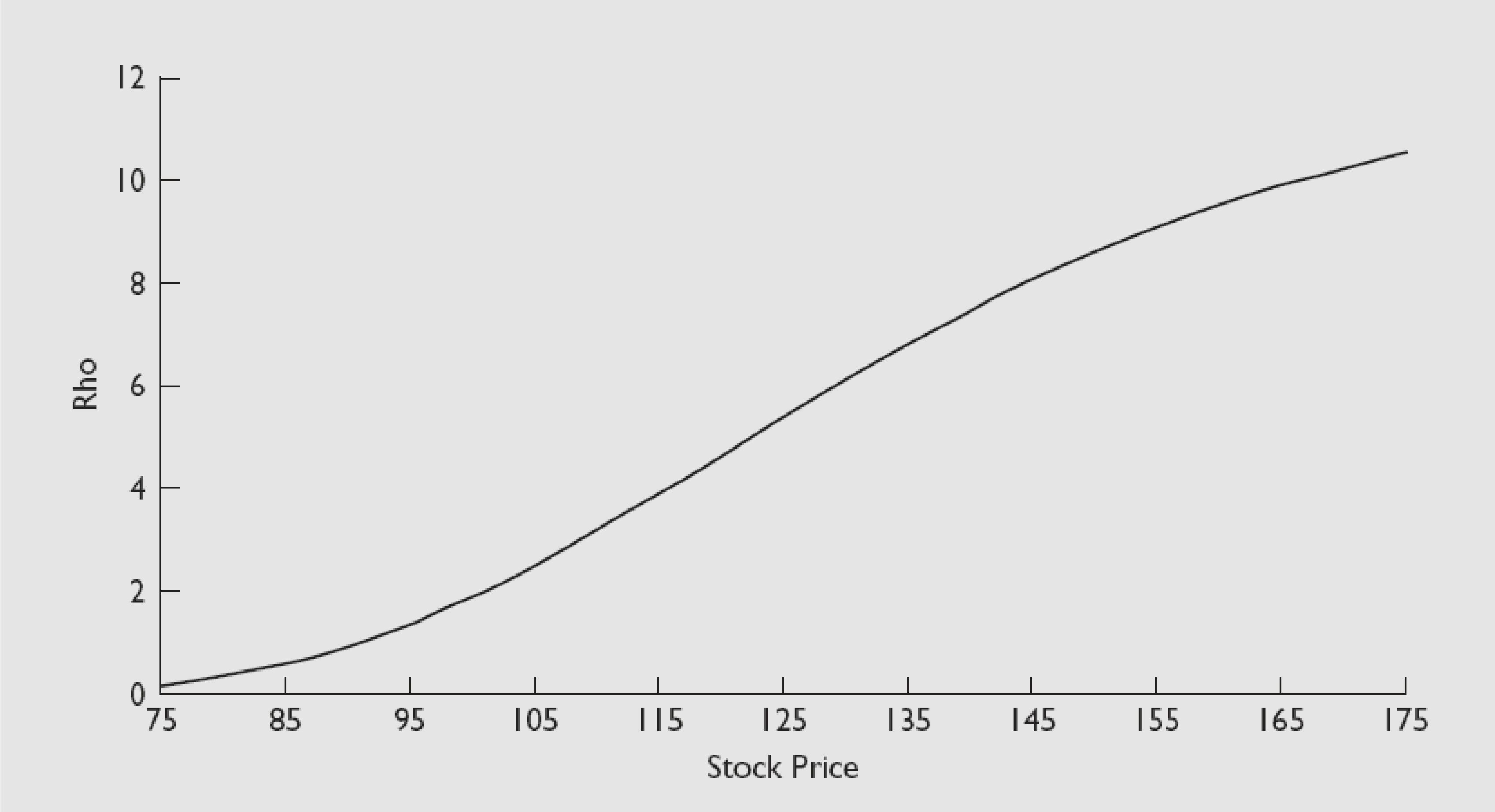

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FIGURE 5.13: The Call Price as a Function of the Volatility DCRB June 125; $S_0 = 125.94$, $r_c = 0.0446$, $T = 0.0959$

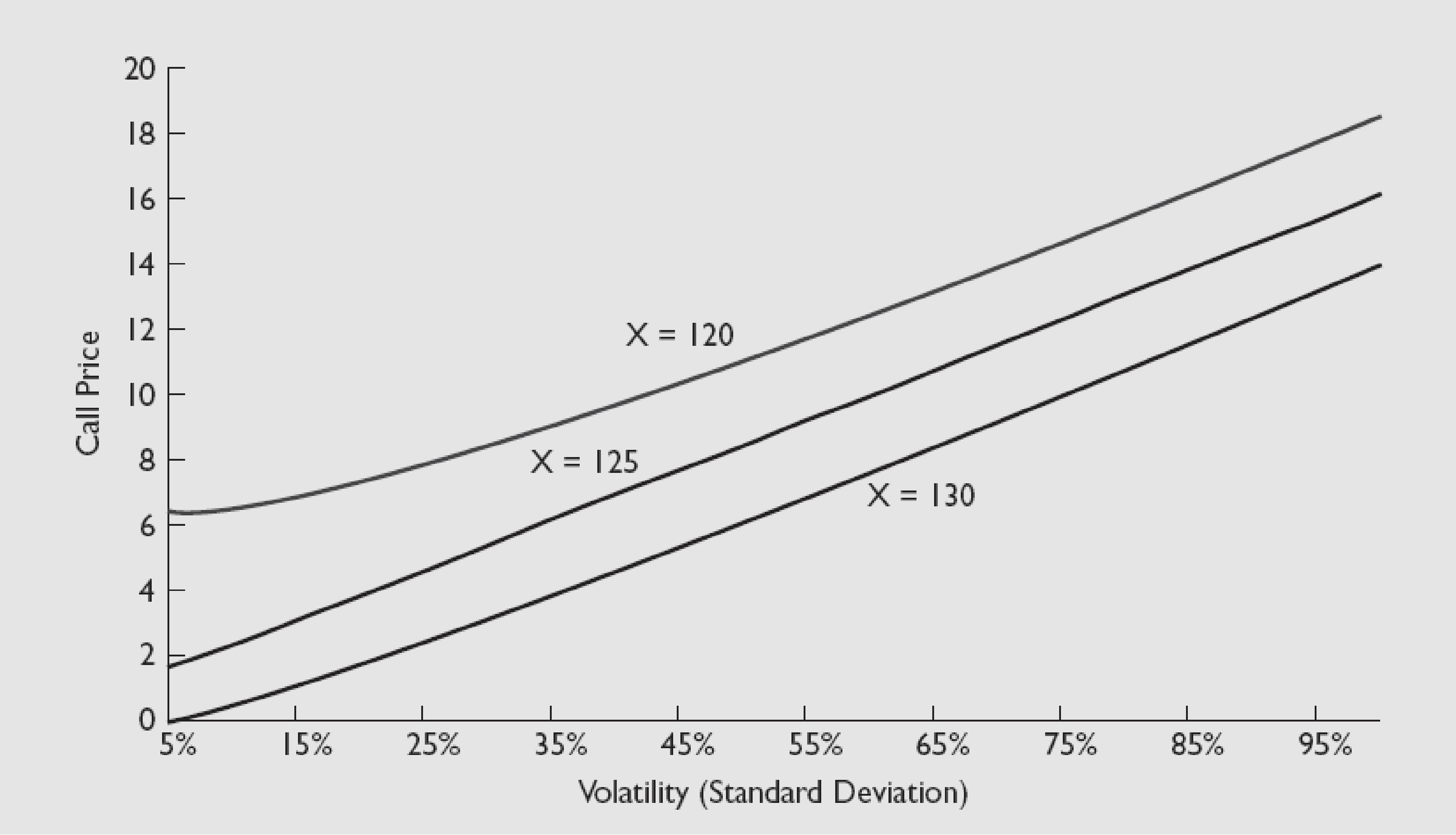

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FIGURE 5.14 : The Vega of a Call as a Function of the Stock Price DCRB June 125; $r_c = 0.0446$, $T = 0.0959$, $\sigma = 0.83$

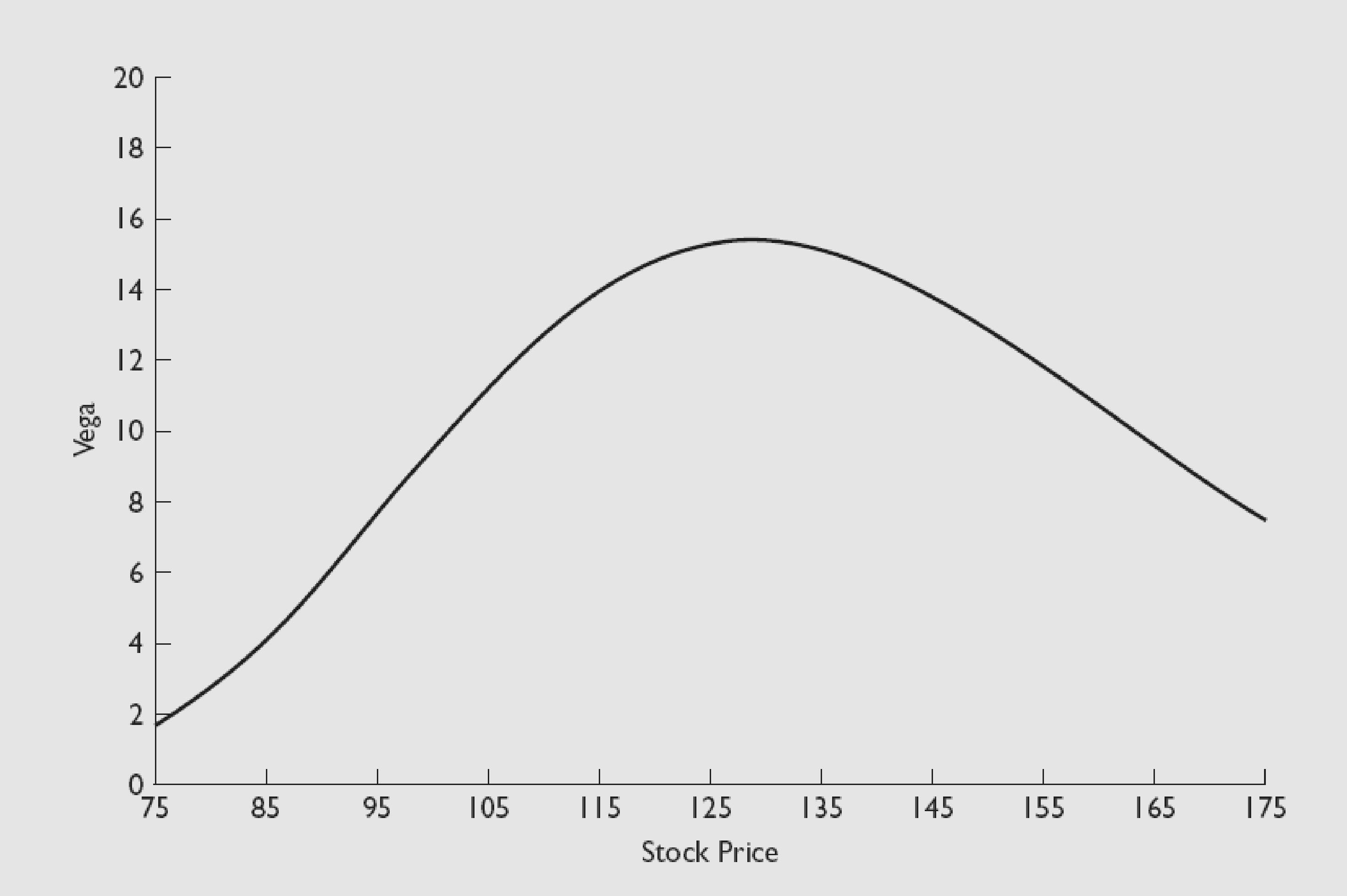

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FIGURE 5.15: The Call Price as a Function of the Time in Days to Expiration DCRB June 125; S_0 = 125.94, r_c = 0.0446, T = 0.0959, σ = 0.83

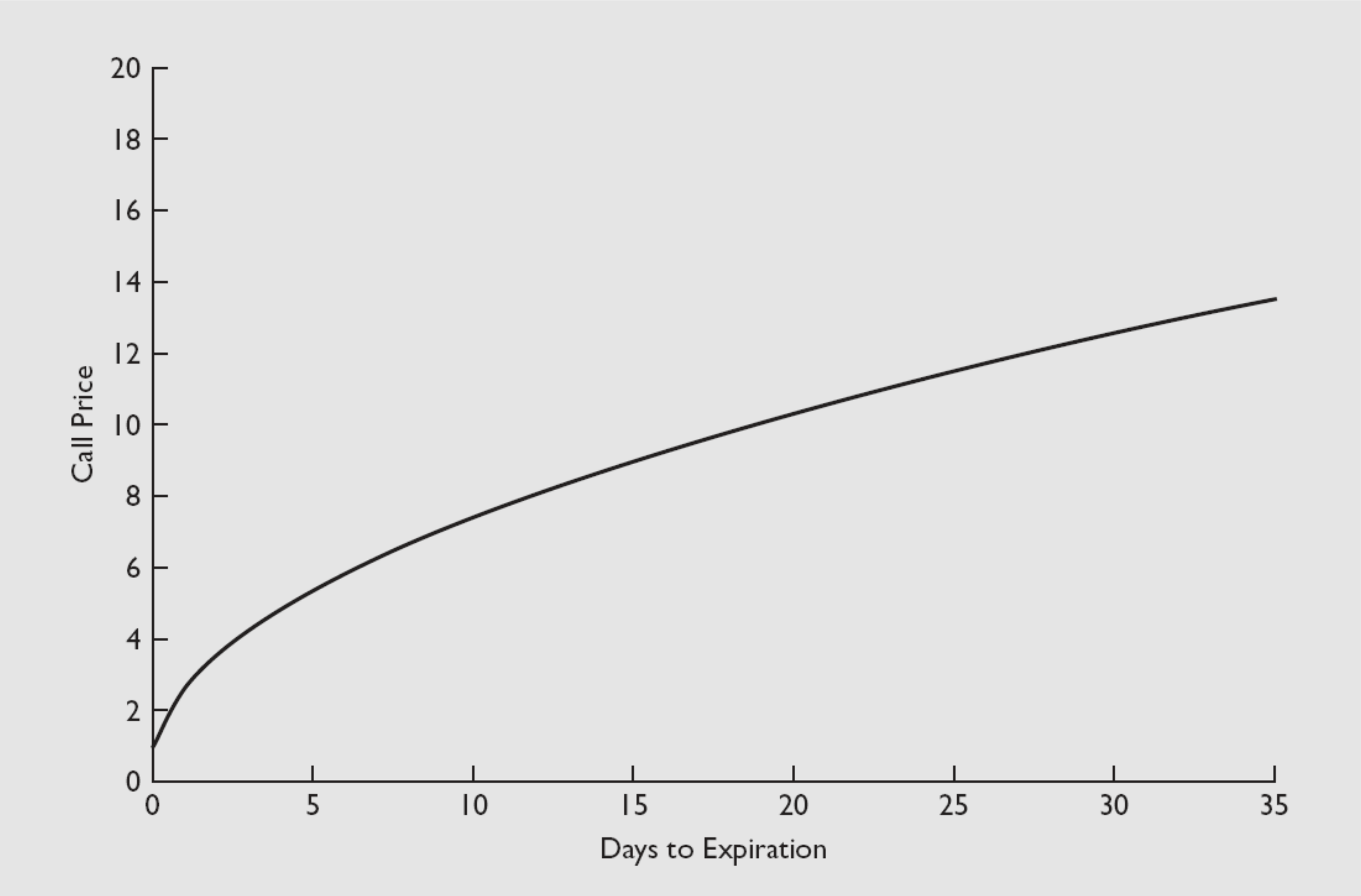

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FIGURE 5.16: The Theta of a Call as a Function of the Stock Price DCRB June 125; r_c = 0.0446, T = 0.0959, σ = 0.83

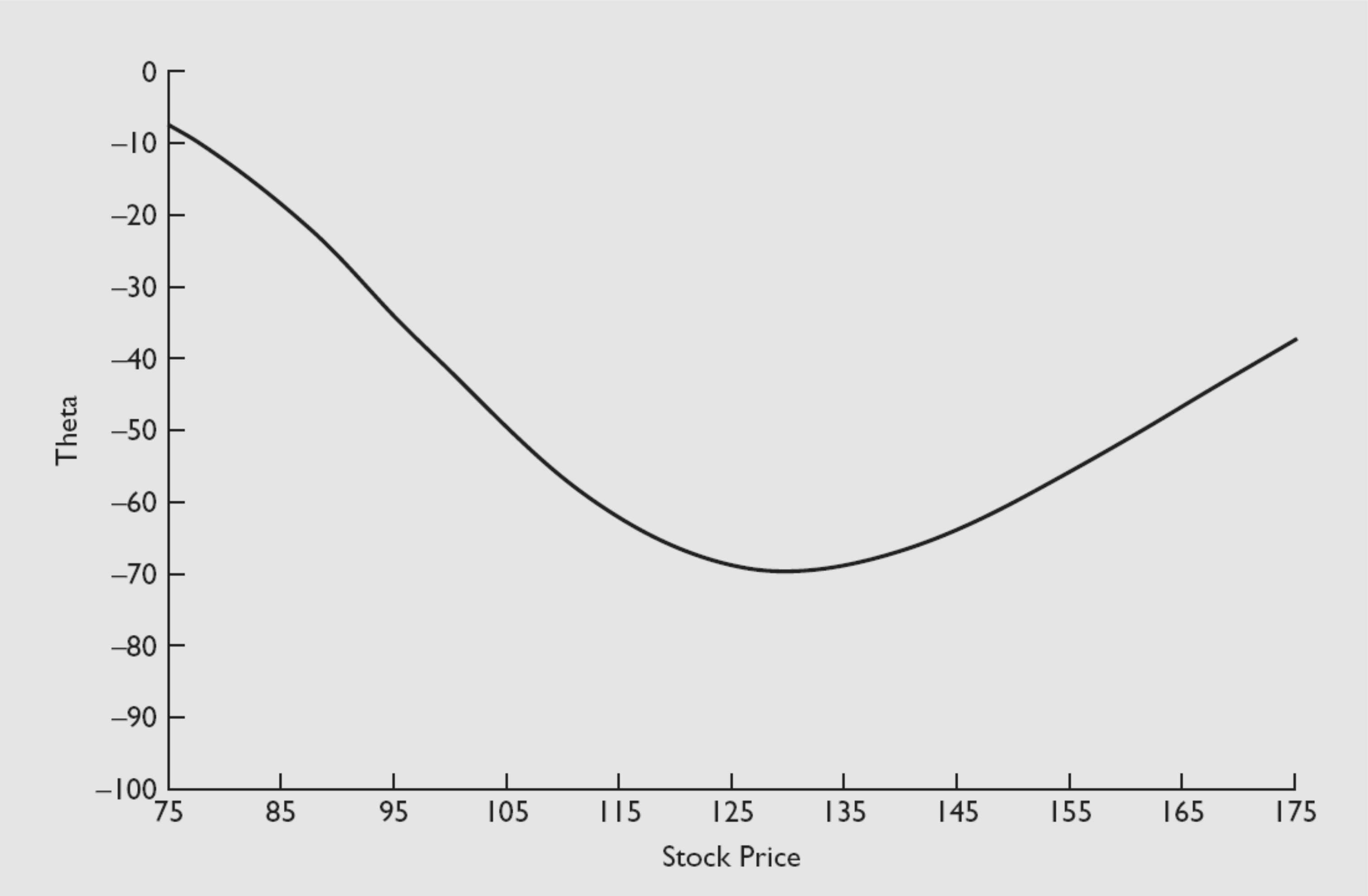

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TABLE 5.3: CALCULATING THE BLACK–SCHOLES–MERTON PRICE WHEN THERE ARE KNOWN DISCRETE DIVIDENDS (1 of 2)

$S_0 = 125.94$ $X = 125$ $r_c = 0.0446$ $\sigma = 0.83$ $T = 0.0959$

One dividend of 2, ex-dividend in 21 days (May 14 to June 4)

1. Determine the amounts of the dividend(s).

$$D_t = \$2$$

2. Determine the time(s) to the ex-dividend date(s).

$$21 \text{ days, so } t = 21/365 = 0.0575$$

3. Subtract the present value of the dividend(s) from the stock price to obtain S'_0.

$$\$125.94 - 2e^{-0.0446(0.0575)} = \$123.9451$$

4. Compute d_1 using S'_0 in place of S_0.

(To continued)

TABLE 5.3: CALCULATING THE BLACK–SCHOLES–MERTON PRICE WHEN THERE ARE KNOWN DISCRETE DIVIDENDS (2 of 2)

$$d_1 = \frac{\ln(123.9451/125) + [0.0446 + (0.83)^2/2]0.0959}{0.83\sqrt{0.0959}} = 0.1122$$

5. Compute d_2.

$$d_2 = 0.1122 - 0.83\sqrt{0.0959} = -0.1449$$

6. Look up $N(d_1)$.

$$N(0.11) = 0.5438$$

7. Look up $N(d_2)$.

$$N(-0.14) = 1 - N(0.14) = 1 - 0.5557 = 0.4443$$

8. Plug into formula for C, using S_0' in place of S_0.

$$C = 123.9451(0.5438) - 125e^{-0.0446(0.0959)}(0.4443) = 12.10$$

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TABLE 5.4: CALCULATING THE BLACK–SCHOLES–MERTON PRICE WHEN THERE ARE CONTINUOUS DIVIDENDS (1 of 2)

S_0 = \$125.94 X = 125 r_c = 0.0446 σ = 0.83 T = 0.0959 δ_c = 0.04

1. Compute the stock price minus the present value of the dividends as $S_0' = S_0 e^{-\delta_e T}$.

$$\$125.94e^{-0.04(0.0959)} = \$125.4578$$

2. Compute d_1 using S_0' in place of S_0.

$$d_1 = \frac{\ln(125.4578/125) + [0.0446 + (0.83)^2/2]0.0959}{0.83\sqrt{0.0959}} = 0.1594$$

3. Compute d_2.

$$d_2 = 0.1594 - 0.83\sqrt{0.0959} = -0.0976$$

4. Look up $N(d_1)$.

$$N(0.16) = 0.5636$$

5. Look up $N(d_2)$.

$$N(-0.10) = 1 - N(0.10) = 1 - 0.5398 = 0.4602$$

(To continued)

TABLE 5.4: CALCULATING THE BLACK–SCHOLES–MERTON PRICE WHEN THERE ARE CONTINUOUS DIVIDENDS (2 of 2)

6. Plug into formula for C, using S_0' in place of S_0.

$$C = 125.4578(0.5636) - 125e^{-0.0446(0.0959)}(0.4602) = 13.43$$

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TABLE 5.5: THE EARLY EXERCISE DECISION

S_0 = 125.94	X = 120	r_c = 0.0446 σ = 0.83	T = 0.0959
DIVIDEND	**VALUE IF EXERCISED[1]**	**VALUE WHEN STOCK GOES EX-DIVIDEND[2]**	**EXERCISE?**
0	5.94	16.03	No
5	5.94	13.03	No
10	5.94	10.35	No
15	5.94	7.99	No
20	5.94	5.98	No
25	5.94	4.32	Yes

[1]The intrinsic value before the stock goes ex-dividend.

[2]Computed using the Black-Scholes-Merton model, with an ex-dividend stock price of $125.94 minus the dividend.

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TABLE 5.6: ESTIMATING THE HISTORICAL VOLATILITY OF DCRB

DATE	PRICE	r_t	r_t^c	$(r_t^c - \bar{r}_t^c)^2$
February 17	80.19	—	—	—
February 18	86.50	0.0787	0.0757	0.004650
February 19	88.00	0.0173	0.0172	0.000093
February 22	87.56	−0.0050	−0.0050	0.000158
February 23	87.19	−0.0042	−0.0042	0.000139
February 24	88.94	0.0201	0.0199	0.000152
February 25	89.56	0.0070	0.0069	0.000000
February 26	86.69	−0.0320	−0.0326	0.001610
March 1	87.06	0.0043	0.0043	0.000011
March 2	86.25	−0.0093	−0.0093	0.000286
May 10	138.44	−0.0212	−0.0214	0.000841
May 11	132.63	−0.0420	−0.0429	0.002543
May 12	125.25	−0.0556	−0.0573	0.004200
Totals			0.4459	0.182817

$\bar{r}^c = \frac{0.4459}{59} = 0.0076 \quad \sigma = \sqrt{\frac{0.182817}{58}} = 0.056143 \quad \text{Annualized } \sigma = 0.056143\sqrt{250} = 0.8877$

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FIGURE 5.17: Calculating the Implied Volatility

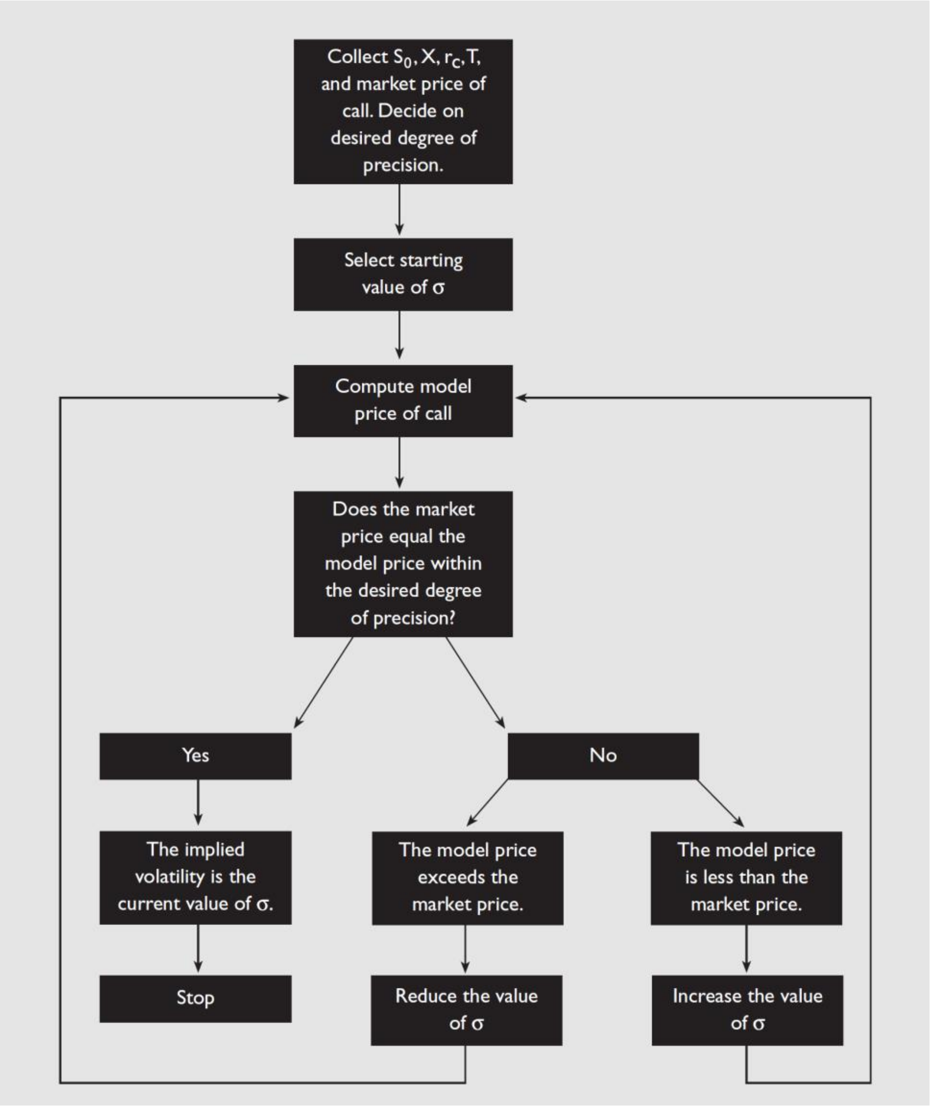

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TABLE 5.7: IMPLIED VOLATILITIES OF DCRB CALLS

EXERCISE PRICE	EXPIRATION: MAY	EXPIRATION: JUNE	EXPIRATION: JULY
120	0.76	0.79	0.85
125	0.75	0.83	0.85
130	0.76	0.83	0.85

S_0 = 125.94, r_c= 0.0447 (May), 0.0446 (June), 0.0453 (July)

T = 0.0192 (May), 0.0959 (June), 0.1726(July)

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FIGURE 5.18: The Term Structure of Implied Volatility DCRB Calls; r_c = 0.0447 (May), 0.0446 (June), 0.0453 (July), T = 0.0192 (May), 0.0959 (June), 0.1726 (July), σ = 0.83

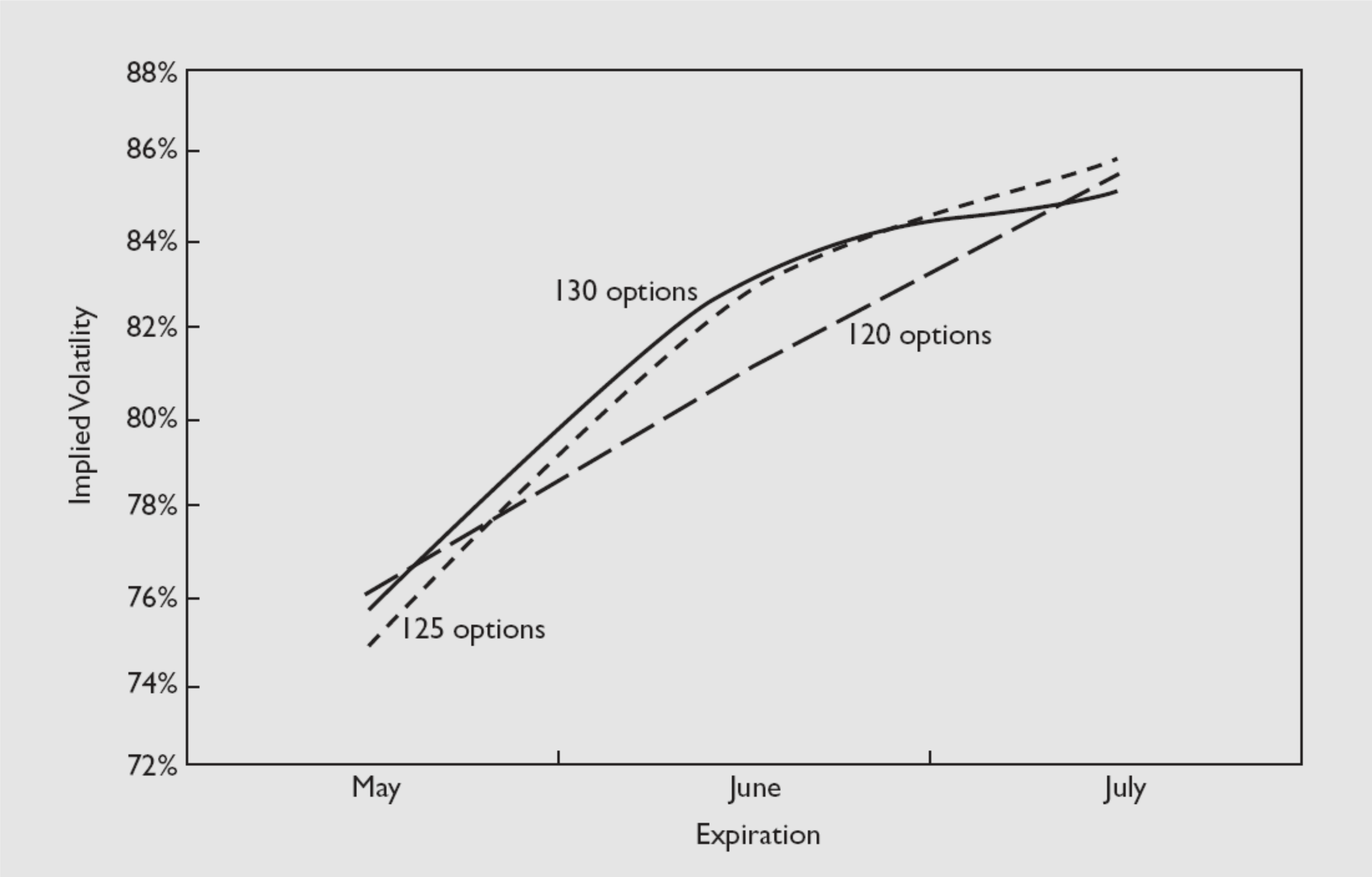

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FIGURE 5.19: The Implied Volatility Smile/Skew DCRB June and July Calls

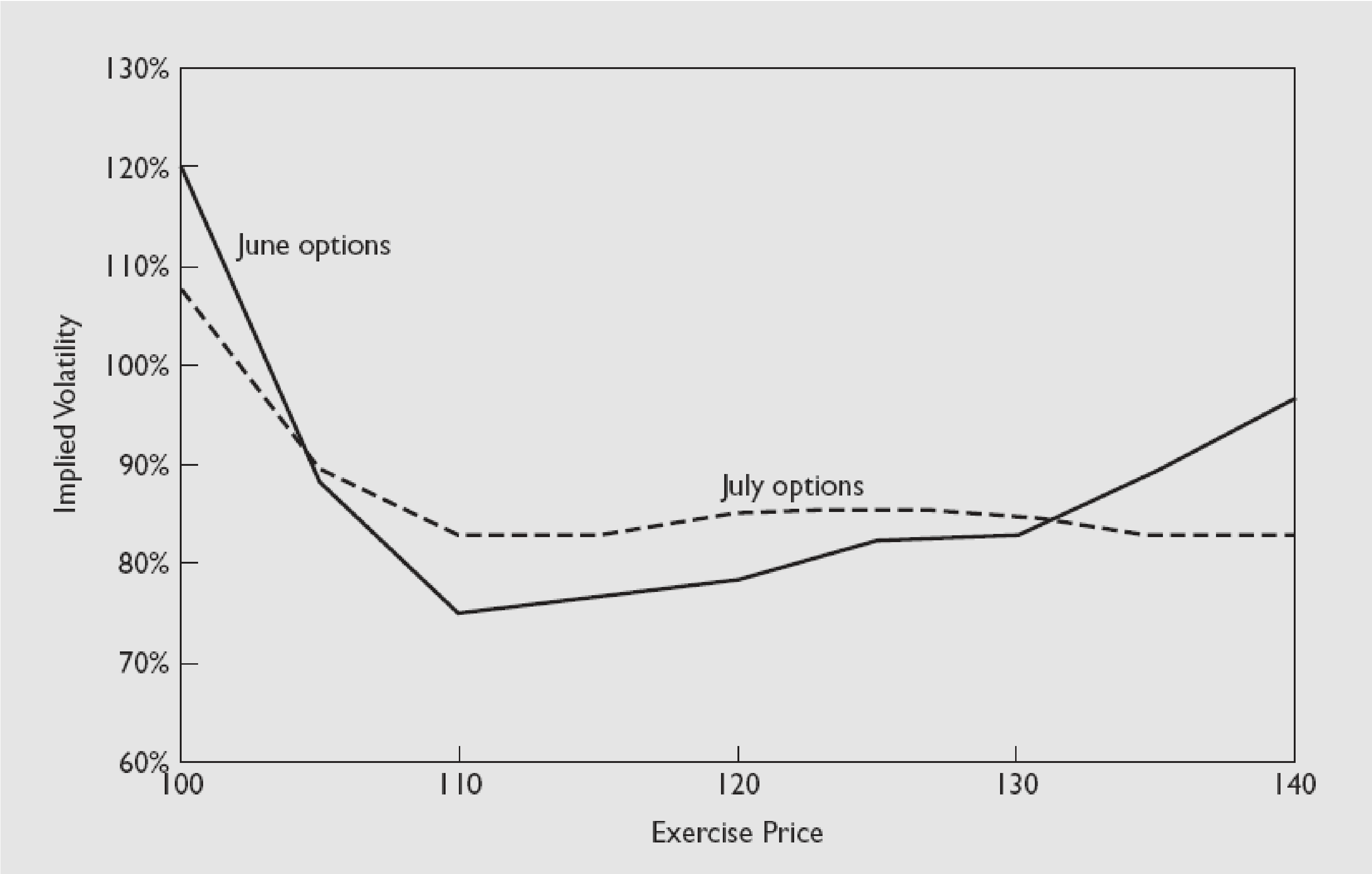

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FIGURE 5.20: The Chicago Board Options Exchange's VIX Index (Implied Volatility of the S&P 500 Index)

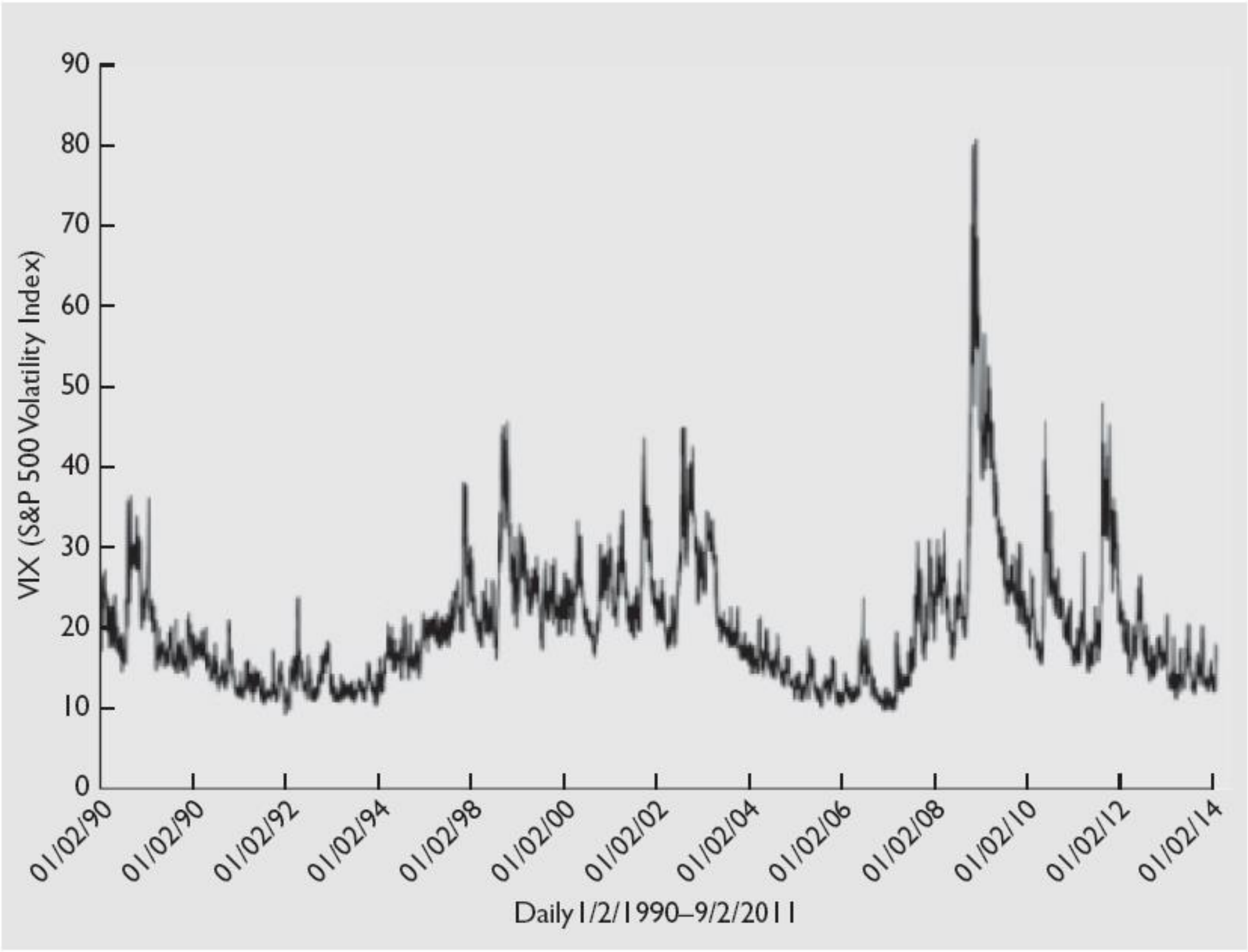

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TABLE 5.8: THE EFFECTS OF VARIABLES ON THE BLACK–SCHOLES–MERTON PUT OPTION PRICE (1 of 3)

The results for put options are obtained by using the put-call parity relationship,

$$P = C - S_0 + Xe^{-r_c T}.$$

Stock Price

The relationship between the stock price and the put price is called the put **delta.**

$$\text{Put } \mathit{Delta} = N(d_1) - 1$$

Because $N(d_1)$ is between 0 and 1, the put delta is negative. This simply means that a put option moves in an opposite direction to the stock. A delta neutral position with stock and puts will require long positions in both stock and puts. The relationship between the put delta and the stock price is called the put *gamma*.

$$\text{Put}\,\mathit{Gamma} = \frac{e^{-d_1^2/2}}{S_0 \sigma \sqrt{2\pi T}}$$

Notice that the put gamma is the same as the call gamma. Like a call, the gamma is highest when the put is at-the-money.

(To continued)

TABLE 5.8: THE EFFECTS OF VARIABLES ON THE BLACK–SCHOLES–MERTON PUT OPTION PRICE (2 of 3)

Exercise Price

For a difference in the exercise price, the put price will change by $e^{-r_c T}\left[1-N(d_2)\right]$.

Because both of the terms are positive, the overall expression is positive.

Risk-Free Rate

The relationship between the put price and the risk-free rate is called the put *rho*.

$$\text{Put } Rho = -T\, Xe^{-r_c T}\left[1-N(d_2)\right]$$

This expression is always negative, meaning that the put price moves inversely to interest rates.

Volatility

The relationship between the put price and the volatility is called the *vega*.

$$\text{Put } Vega = \frac{S_0\sqrt{T}e^{-d_1^2/2}}{\sqrt{2\pi}}$$

This is the same expression as the vega for a call. Like a call, a put is highly sensitive to the volatility.

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TABLE 5.8: THE EFFECTS OF VARIABLES ON THE BLACK–SCHOLES–MERTON PUT OPTION PRICE (3 of 3)

Time to Expiration

The relationship between the put price and the time to expiration is called the *theta.*

$$\text{Put } \textit{Theta} = -\frac{S_0 \sigma e^{-d_1^2/2}}{2\sqrt{2\pi T}} + r_c X e^{-r_c T}\left[1 - N(d_2)\right]$$

This expression can be positive or negative, a point we covered in Chapter 3. A European put can either increase or decrease with time. The tendency to increase occurs because there is more time for the stock to move favorably. The tendency to decrease occurs because the additional time means waiting longer to receive the exercise price at expiration.

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TABLE 5.9: DAILY ADJUSTED DELTA HEDGE: S_0 = 125.94, X = 125, r_c = 0.0446, T = 0.0959, σ = 0.83, 1,000 SHORT CALLS (1 of 2)

BEGINNING-OF-DAY HOLDINGS			END-OF-DAY VALUES (BEFORE ADJUSTMENTS)					
DAY (1)	SHARES (2)	CALLS (3)	$ OF BONDS (4)	STOCK PRICE (5)	CALL PRICE (6)	PORTFOLIO VALUE (7)	TARGET PORTFOLIO VALUE (8)	NEW CALL TRADES (9)
0	0	0	0	125.9400	13.5533	0	58107	−1000
1	569	−1000	0	120.4020	10.4078	58101	58114	0
2	498	−1000	8550	126.2305	13.3358	58077	58121	0
3	572	−1000	−792	129.7259	15.2113	58200	58128	0
4	614	−1000	−6241	122.6991	11.0082	58088	58135	0
5	524	−1000	4803	116.8313	7.9826	58040	58142	0
6	441	−1000	14501	111.9864	5.8418	58046	58149	0
7	368	−1000	22679	104.3922	3.3160	57780	58156	0
8	257	−1000	34271	95.8023	1.4834	57409	58163	0
9	147	−1000	44815	97.6316	1.6663	57500	58170	0
10	161	−1000	43453	101.2905	2.2093	57552	58178	0
11	199	−1000	39609	107.7229	3.6022	57444	58185	0
12	281	−1000	30779	108.7139	3.7173	57611	58192	0
13	290	−1000	29805	100.3039	1.6632	57230	58199	0
14	167	−1000	42147	99.4745	1.4164	57343	58206	0
15	150	−1000	43844	106.1533	2.5476	57219	58213	0
16	232	−1000	35143	97.1141	0.9052	56769	58220	0
17	110	−1000	46997	94.3615	0.5666	56810	58227	0
18	78	−1000	50023	86.0927	0.1300	56608	58234	0
19	24	−1000	54678	88.4161	0.1625	56638	58242	0
20	29	−1000	54243	93.3283	0.3112	56638	58249	0
21	50	−1000	52289	90.6496	0.1604	56662	58256	0
22	30	−1000	54109	92.6696	0.1857	56703	58263	0
23	34	−1000	53745	92.2053	0.1325	56747	58270	0
24	26	−1000	54489	97.6458	0.2873	56741	58277	0
25	51	−1000	52054	101.2303	0.4257	56791	58284	0
26	73	−1000	49833	106.4213	0.7980	56804	58291	0
27	123	−1000	44518	110.5062	1.2135	56897	58299	0
28	175	−1000	38776	110.4197	0.9792	57120	58306	0
29	155	−1000	40990	108.7435	0.5619	57283	58313	0
30	106	−1000	46324	109.4487	0.4535	57472	58320	0
31	95	−1000	47533	111.5737	0.4654	57668	58327	0
32	104	−1000	46535	115.9403	0.7627	57830	58334	0
33	169	−1000	39004	122.7231	2.0457	57698	58341	0
34	396	−1000	11147	120.9295	0.6922	58343	58348	0
35	231	−1000	31104	124.0621	0.0002	59762	58356	0

(To continued)

TABLE 5.9: DAILY ADJUSTED DELTA HEDGE: S_0 = 125.94, X = 125, r_c = 0.0446, T = 0.0959, σ = 0.83, 1,000 SHORT CALLS (2 of 2)

ADJUSTED NO. OF SHARES (10)	NO. OF SHARES BOUGHT OR SOLD (11)	$ OF BONDS PURCHASED OR SOLD (12)	TIME TO EXPIRATION (13)	END-OF-DAY OPTION CALCULATIONS d_1 (14)	$N(d_1)$ (15)	d_2 (16)	$N(d_2)$ (17)	PORTFOLIO DELTA (18)
569	569	0	0.0959	0.1743	0.5692	−0.0827	0.4670	0.1868
498	−71	8549	0.0932	−0.0049	0.4981	−0.2582	0.3981	−0.0570
572	74	−9341	0.0904	0.1802	0.5715	−0.0694	0.4723	0.4981
614	42	−5448	0.0877	0.2898	0.6140	0.0440	0.5176	−0.0120
524	−90	11043	0.0849	0.0598	0.5238	−0.1821	0.4278	0.1536
441	−83	9697	0.0822	−0.1496	0.4405	−0.3876	0.3492	0.4638
368	−73	8175	0.0795	−0.3377	0.3678	−0.5717	0.2838	0.2222
257	−111	11588	0.0767	−0.6538	0.2566	−0.8837	0.1884	0.3818
147	−110	10538	0.0740	−1.0509	0.1467	−1.2766	0.1009	0.3436
161	14	−1367	0.0712	−0.9903	0.1610	−1.2119	0.1128	−0.0064
199	38	−3849	0.0685	−0.8455	0.1989	−1.0627	0.1440	0.0803
281	82	−8833	0.0658	−0.5787	0.2814	−0.7915	0.2143	−0.4097
290	9	−978	0.0630	−0.5523	0.2904	−0.7606	0.2234	−0.3815
167	−123	12337	0.0603	−0.9650	0.1673	−1.1688	0.1212	−0.2715
150	−17	1691	0.0575	−1.0348	0.1504	−1.2339	0.1086	−0.3890
232	82	−8705	0.0548	−0.7314	0.2323	−0.9257	0.1773	−0.2782
110	−122	11848	0.0521	−1.2259	0.1101	−1.4153	0.0785	−0.1154
78	−32	3020	0.0493	−1.4213	0.0776	−1.6056	0.0542	0.3802
24	−54	4649	0.0466	−1.9803	0.0238	−2.1595	0.0154	0.1674
29	5	−442	0.0438	−1.8942	0.0291	−2.0680	0.0193	−0.0998
50	21	−1960	0.0411	−1.6413	0.0504	−1.8096	0.0352	−0.3658
30	−20	1813	0.0384	−1.8846	0.0297	−2.0472	0.0203	0.2580
34	4	−371	0.0356	−1.8218	0.0342	−1.9785	0.0239	−0.2390
26	−8	738	0.0329	−1.9367	0.0264	−2.0872	0.0184	−0.3930
51	25	−2441	0.0301	−1.6323	0.0513	−1.7765	0.0378	−0.3037
73	22	−2227	0.0274	−1.4574	0.0725	−1.5948	0.0554	0.4937
123	50	−5321	0.0247	−1.1607	0.1229	−1.2911	0.0983	0.1260
175	52	−5746	0.0219	−0.9333	0.1753	−1.0562	0.1454	−0.3251
155	−20	2208	0.0192	−1.0138	0.1553	−1.1288	0.1295	−0.3333
106	−49	5328	0.0164	−1.2487	0.1059	−1.3552	0.0877	0.1158
95	−11	1204	0.0137	−1.3123	0.0947	−1.4095	0.0794	0.2860
104	9	−1004	0.0110	−1.2581	0.1042	−1.3450	0.0893	−0.1790
169	65	−7536	0.0082	−0.9568	0.1693	−1.0321	0.1510	−0.3413
396	227	−27858	0.0055	−0.2642	0.3958	−0.3257	0.3723	0.1939
231	−165	19953	0.0027	−0.7361	0.2308	−0.7796	0.2178	0.1717
2	−229	28410	0.0000	−2.9288	0.0017	−2.9314	0.0017	0.2986

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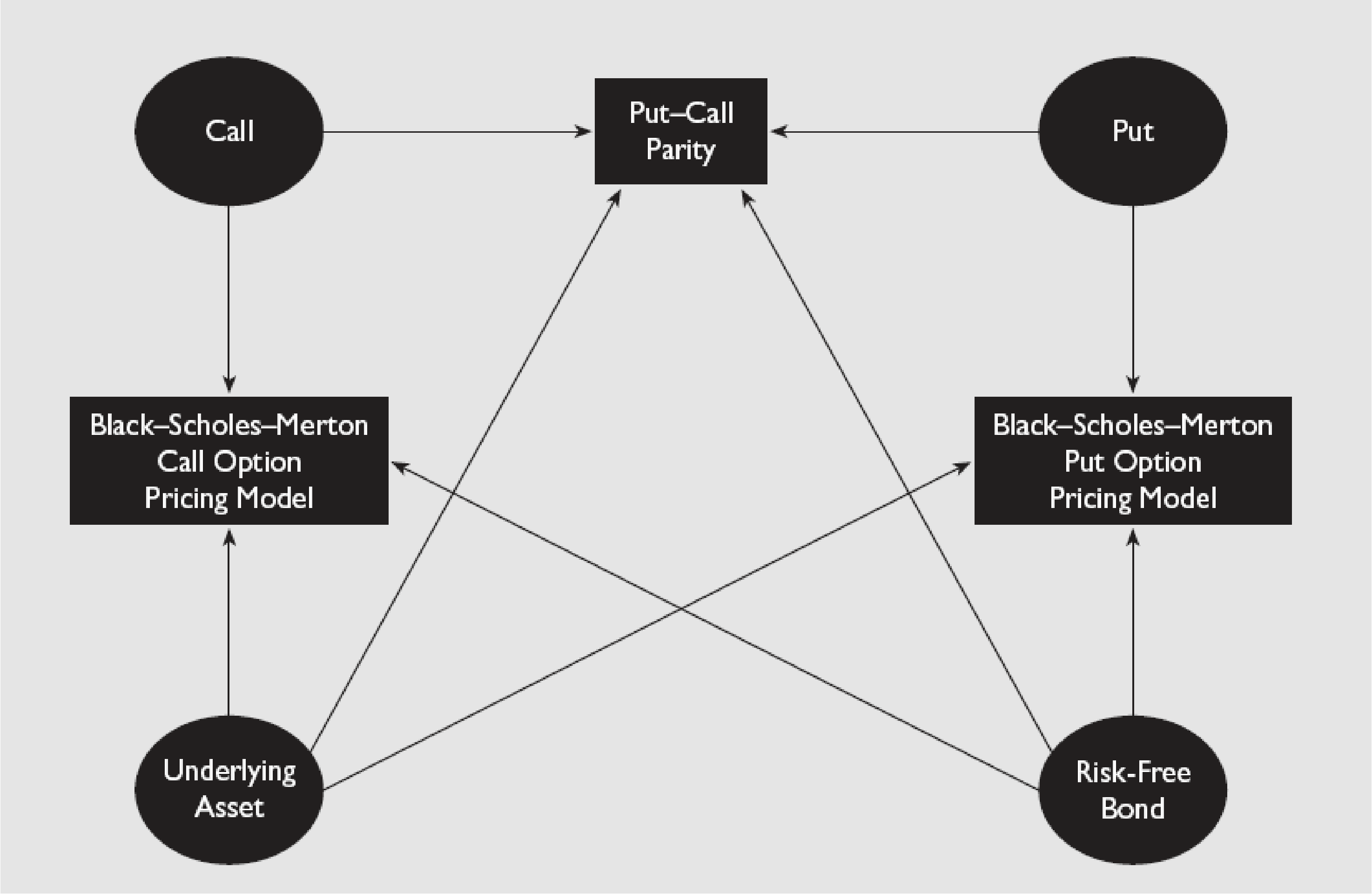
FIGURE 5.21: The Linkage between Calls, Puts, the Underlying Asset, and Risk-Free Bonds



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